SEIDEL ELEMENTS AND POTENTIAL FUNCTIONS OF HOLOMORPHIC DISC COUNTING

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ABSTRACT. Let M be a symplectic manifold equipped with a Hamiltonian circle action and let L be an invariant Lagrangian submanifold of M. We study the problem of counting holomorphic *disc sections* of the trivial M-bundle over a disc with boundary in L through degeneration. We obtain a conjectural relationship between the potential function of L and the Seidel element associated to the circle action. When applied to a Lagrangian torus fibre of a semi-positive toric manifold, this degeneration argument reproduces a conjecture (now a theorem) of Chan-Lau-Leung-Tseng [7, 8] relating certain correction terms appearing in the Seidel elements with the potential function.

1. INTRODUCTION

Let M be a symplectic manifold with a Hamiltonian circle action. Seidel [25] constructed an invertible element of the quantum cohomology of M by counting pseudo-holomorphic sections of the associated M-bundle E over S^2:

$$E = (M \times S^3)/S^1$$

where S^1 acts by the diagonal action and $S^3 \to S^2$ is the Hopf fibration. Seidel elements have been used to detect essential loops in the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms. McDuff-Tolman [24] used them to verify Batyrev's presentation of quantum cohomology rings for toric varieties.

In the previous paper [19], we computed Seidel elements of semi-positive toric manifolds and found that they are closely related to Givental's mirror transformation [17]. Chan-Lau-Leung-Tseng [7] conjectured that certain correction terms appearing in our computation of Seidel elements determine the potential function of a Lagrantian torus fibre. The potential function here is given by counting holomorphic discs with boundary in a Lagrangian torus fibre and is thought of as a mirror of the toric variety. The conjecture was proved by themselves [8] in a recent preprint. In this paper, we propose an alternative approach which relates Seidel elements and potential functions via *degeneration*. Our method should apply to a general symplectic manifold M with a Hamiltonian S^1-action and an invariant Lagrangian.

We assume that M is a smooth projective variety, equipped with a $\mathbb{C}^\times$-action and an S^1-invariant Kähler form ω. Let L be an S^1-invariant Lagrangian submanifold of M. Let $\mathcal{M}_1(\beta)$ denote the moduli space of genus-zero bordered stable holomorphic maps from $(\Sigma, \partial\Sigma)$ to (M, L) with one boundary marking and representing $\beta \in H_2(M, L)$. By the fundamental work of Fukaya-Oh-Ohta-Ono [12, 15], $\mathcal{M}_1(\beta)$ is compact and carries a Kuranishi structure with boundary and corner. Let β be a class of Maslov index two. Under certain assumptions

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(see §2.1), the virtual fundamental *chain* of $\mathcal{M}_1(\beta)$ is a *cycle* of dimension $\dim_{\mathbb{R}} L$ and one can define the *open Gromov-Witten invariant* $n_\beta \in \mathbb{Q}$ by

$$\mathrm{ev}_*[\mathcal{M}_1(\beta)]^{\mathrm{vir}} = n_\beta [L]$$

where $\mathrm{ev}\colon \mathcal{M}_1(\beta) \to L$ is the evaluation map. The potential function W is

$$W = \sum_{\beta \in H_2(M,L):\mu(\beta)=2} n_\beta z^\beta.$$

The idea of degeneration is that instead of counting discs in (M, L), we consider the problem of counting *disc sections* of the trivial bundle $M \times \mathbb{D} \to \mathbb{D}$ with boundary in $L \times S^1$. Then we degenerate the target $M \times \mathbb{D}$ to the union $E \cup_M (M \times \mathbb{D})$. From this geometry we expect the following *degeneration formula* (see §3.3 for details):

$$\varphi_* \,\mathrm{ev}_*[\mathcal{M}_1(\hat{\beta})]^{\mathrm{vir}} = \sum_{r(\hat{\beta})=\sigma+\hat{\alpha}} \mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}} \tag{1}$$

if the both-hand sides carry virtual fundamental *cycles*, instead of *chains*. Here $\hat{\beta} \in H_2(M \times \mathbb{D}, L \times S^1)$ denotes a disc section class corresponding to $\beta \in H_2(M, L)$ and $\mathcal{M}_1(\hat{\beta})$ is the corresponding moduli space of disc sections. The summation in the right-hand side is taken over all possible decompositions $\sigma + \hat{\alpha}$ of the class $\hat{\beta}$ into a section class σ of E and another disc section class $\hat{\alpha}$ under the degeneration. Also $\mathcal{M}_{\mathrm{S}}(\sigma)$ is a moduli space of holomorphic sections of E in the class σ, which is relevant to the Seidel element. This formula relates disc counts of different boundary types; the boundary classes $\partial\alpha$ and $\partial\beta$ from the both-hand sides differ exactly by the S^1-action.

The degeneration formula predicts a relationship between the Seidel element of the S^1-action and the potential function W. We need the following conditions in order to extract meaningful information from the formula (1):

(i) $\mathcal{M}_1(\beta)$ is empty for all $\beta \in H_2(M, L)$ with $\mu(\beta) \leq 0$.
(ii) The maximal fixed component $F_{\max} \subset M$ of the $\mathbb{C}^\times$-action (see §2.2) is of complex codimension one and the $\mathbb{C}^\times$-weight on the normal bundle is -1.
(iii) $c_1(M)$ is semi-positive.
(iv) $\mathrm{ev}(\mathcal{M}_{\mathrm{S}}(\sigma))$ is disjoint from L for all $\sigma \in H_2^{\mathrm{sec}}(E)$ such that $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle = -1$.

Theorem 1.1 (Corollary 3.21)**.** *Assume that M is simply-connected and L is connected. Assume that the degeneration formula* (1) *holds (see Conjecture 3.17 for a precise formulation) and that the above conditions (i)–(iv) are satisfied. Then*

$$z^{\alpha_0} = \langle \widehat{S}^{(2)}, dW \rangle + \widetilde{S}^{(0)}$$

holds in a certain "open" Novikov ring Λ^{op} (see §2.1), where

- $\alpha_0 \in H_2(M, L)$ *is the maximal disc class defined by rotating a path connecting L and $F_{\max}$ by the S^1-action (see §3.2);*
- $dW = \sum_{\mu(\beta)=2} \beta \otimes n_\beta z^\beta$ *is the logarithmic derivative of W;*
- $\widetilde{S} = \widetilde{S}^{(0)} + \widetilde{S}^{(2)}$ *is the Seidel element associated to the S^1-action and $\widetilde{S}^{(i)} \in H^i(M) \otimes \Lambda$ (Λ is the "closed" Novikov ring in Remark 2.9);*
- $\widehat{S}^{(2)} \in H^2(M, L) \otimes \Lambda$ *is a lift of $\widetilde{S}^{(2)}$ (see Definition 3.19).*

In particular,

$$\mathrm{KS}(\widetilde{S}) = [z^{\alpha_0}]$$

holds in a certain Jacobi algebra of W, where KS *denotes the Kodaira-Spencer mapping (see the end of §3.3.3).*

In the second half of the paper, we apply these to a semi-positive toric manifold X and calculate the potential function of a Lagrangian torus fibre $L \subset X$. In toric case, the potential function can be regarded as a function on the moduli space $\mathfrak{M}_{\mathrm{opcl}}$ of Lagrangian torus fibres L together with complexfied Kähler classes $-\omega + iB$ and lifts $h \in H^2(X, L; U(1))$ of $\exp(iB)$ (see §4.2.1, h defines a $U(1)$-local system on L when $B = 0$). The potential function is of the form:

$$W = w_1 + \cdots + w_m$$

with $w_i = f_i(q) z_i$, where $f_i(q) \in \Lambda$ is the *correction term* defined by

$$f_i(q) = \sum_{d \in H_2(X;\mathbb{Z}) : \langle c_1(X), d\rangle = 0} n_{\beta_i + d} q^d.$$

Each term w_i corresponds to a prime toric divisor $D_i \subset X$ and arises from disc counting of fixed boundary type $b_i \in H_1(L)$. Applying the degeneration formula, we get:

Theorem 1.2 (Theorem 4.13)**.** *Assume that the degeneration formula* (1) *(Conjecture 3.17) holds for (X, L) equipped with the $\mathbb{C}^\times$-action ρ_j rotating around the prime toric divisor D_j (see §4.3). Let $\widetilde{S}_j \in H^2(X) \otimes \Lambda$ be the Seidel element ρ_j and let $\widehat{S}_j \in H^2(X, L) \otimes \Lambda$ be its lift. Then we have*

$$\langle \widehat{S}_j, dw_k \rangle = \delta_{jk} z_j.$$

In particular we have $\mathrm{KS}(\widetilde{S}_j) = [z_j]$.

We observe in Theorem 4.14 that the degeneration formula reproduces the following conjecture (now a theorem) of Chan-Lau-Leung-Tseng [7, 8].

Theorem 1.3 ([7, Conjecture 4.12], [8, Theorem 1.1])**.** *Let $g_0^{(j)}(y)$, $j = 1, \ldots, m$ be explicit hypergeometric functions in variables $y_1, \ldots, y_r$ ($r = \dim H^2(X)$) given in equation* (37)*. Then we have*

$$f_j(q) = \exp\left(g_0^{(j)}(y)\right)$$

under an explicit change of variables (mirror transformation) of the form $\log q_i = \log y_i + g_i(y)$, $i = 1, \ldots, r$ with $g_i(y) \in \mathbb{Q}[\![y_1, \ldots, y_r]\!]$ and $g_i(0) = 0$.

In [19], we introduced *Batyrev elements* $\widetilde{D}_j$ as mirror analogues of the divisor classes D_j. They satisfy the relations of Batyrev's quantum ring [4] for toric varieties. The hypergeometric functions $g_0^{(j)}(y)$ originally appeared in our computation [19] as the difference between the Seidel and the Batyrev elements:

$$\widetilde{D}_j = \exp\left(g_0^{(j)}(y)\right) \widetilde{S}_j.$$

Hence by Theorem 1.2, $\widetilde{S}_j$ and $\widetilde{D}_j$ correspond respectively to $[z_j]$ and $[w_j]$ under the Kodaira-Spencer mapping (see also [8, Theorem 1.5]).

Finally we discuss briefly the method of Chan-Lau-Leung-Tseng [8]. Their approach is different from ours but is closely related to it. They observed that a holomorphic disc in (X, L) whose boundary class is $b_j \in H_1(L)$ can be completed to a holomorphic *sphere* in the M-bundle E'_j associated to the inverse $\mathbb{C}^\times$-action ρ_j^{-1}. Using this, they identified open Gromov-Witten invariants of (X, L) with certain closed invariants of E'_j. The associated bundle E' of the inverse action also appears in our story as the central fibre $E \cup_M E'$ of the degeneration of the closed manifold $M \times \mathbb{P}^1$ (instead of $M \times \mathbb{D}$) in §3.1.

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2. Preliminaries

In this section, we review a potential function of a Lagrangian submanifold and a Seidel element associated to a Hamiltonian circle action.

2.1. Potential function of a Lagrangian submanifold.

The potential of a Lagrangian submanifold arises as the 0-th operation $\mathfrak{m}_0$ of the corresponding A_∞-algebra in Lagrangian Floer theory of Fukaya-Oh-Ohta-Ono [12]. In this paper, we do not use the full generality of A_∞-formalism developed in [12]; instead we consider potential functions under certain restrictive assumptions.

Let (M, ω) be a closed symplectic manifold and L be a Lagrangian submanifold. For simplicity, we restrict ourselves to the case where M is a smooth projective variety. We assume that L is oriented, relatively-spin and fix a relative spin structure [12, Definition 8.1.2] of L so that the moduli space of bordered stable maps to (M, L) has an oriented Kuranishi structure. Let $\mu\colon H_2(M, L) \to \mathbb{Z}$ denote the Maslov index. It takes values in $2\mathbb{Z}$ since L is oriented.

Let $\mathcal{M}_1(\beta)$ denote the moduli space of stable holomorphic maps from a genus-zero bordered Riemann surface $(\Sigma, \partial\Sigma)$ to (M, L) with one boundary marked point and in the class $\beta \in H_2(M, L)$. This was denoted by $\mathcal{M}_1^{\mathrm{main}}(\beta)$ in [12]. By [12, Proposition 7.1.1] (see also [15, Theorem 15.3]), $\mathcal{M}_1(\beta)$ is compact and equipped with an oriented Kuranishi structure (with boundary and corner) and has virtual dimension $n + \mu(\beta) - 2$, where $n = \dim_{\mathbb{R}} L$. Let $\mathrm{ev}\colon \mathcal{M}_1(\beta) \to L$ denote the evaluation map. Define an open version of Novikov ring Λ^{op} to be the space of all formal power series

$$\sum_{\beta \in H_2(M,L)} c_\beta z^\beta$$

with $c_\beta \in \mathbb{Q}$ such that

$$\sharp \left\{ \beta \,:\, c_\beta \neq 0,\ \textstyle\int_\beta \omega < E \right\} < \infty$$

holds for all $E \in \mathbb{R}$.

Definition 2.1. Assume that $\mathcal{M}_1(\beta)$ is empty for all $\beta \in H_2(M, L)$ with $\mu(\beta) \leq 0$. Then $\mathcal{M}_1(\beta)$ with $\mu(\beta) = 2$ has no boundary and carries a virtual fundamental *cycle* of dimension $n = \dim_{\mathbb{R}} L$ [12, Lemma A.1.32]. We define *open Gromov-Witten invariants* $n_\beta \in \mathbb{Q}$ by

$$\mathrm{ev}_*[\mathcal{M}_1(\beta)]^{\mathrm{vir}} = n_\beta [L]$$

for β with $\mu(\beta) = 2$, where $[L] \in H_n(L)$ is the fundamental class of L. The *potential function* of L is defined to be the formal sum:

$$W = \sum_{\beta \in H_2(M,L):\mu(\beta)=2} n_\beta z^\beta.$$

This is an element of Λ^{op}.

We can decompose W according to boundary classes of discs.

Definition 2.2. Under the same assumption as in Definition 2.1, we write

$$W = \sum_{\gamma \in H_1(L)} W_\gamma$$

with $W_\gamma \in \Lambda^{\mathrm{op}}$ given by

$$W_\gamma := \sum_{\beta \in H_2(M,L):\, \mu(\beta)=2, \partial\beta=\gamma} n_\beta z^\beta.$$

Remark 2.3. The potential function does depend on the choice of a complex structure on M and this is a reason why we restricted to a smooth projective variety M. For example, the Hirzebruch surfaces $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_2$ together with their Lagrangian torus fibres are symplectomorphic to each other, but the potential functions are different. See Auroux [2] for a wall-crossing of disc counting.

2.2. **Seidel elements.** Seidel element is an invertible element of quantum cohomology associated to a loop in the group $\mathrm{Ham}(M, \omega)$ of Hamiltonian diffeomorphisms of a symplectic manifold (M, ω). In this paper we restrict to the case where M is a smooth projective variety equipped with an algebraic $\mathbb{C}^\times$-action. In this case, the associated S^1-action is Hamiltonian and yields a loop in $\mathrm{Ham}(M, \omega)$. We refer the reader to [25, 22, 23] for the original definitions and to [24, 18] for applications in symplectic topology.

Let M be a smooth projective variety, equipped with a $\mathbb{C}^\times$-action.

Definition 2.4. The *associated bundle* of the $\mathbb{C}^\times$-action on M is the M-bundle over $\mathbb{P}^1$

$$E := M \times (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times \to \mathbb{P}^1,$$

where $\mathbb{C}^\times$ acts with the diagonal action $\lambda \cdot (x, (z_1, z_2)) = (\lambda x, (\lambda z_1, \lambda z_2))$.

Remark 2.5. In symplectic geometric terms, the associated bundle is in fact a clutched bundle obtained by gluing two trivial M-bundles over the unit disc, along the boundary, using the action. More precisely,

$$E = (M \times \mathbb{D}_0) \cup_g (M \times \mathbb{D}_\infty)$$

where $\mathbb{D}_0 = \{z \in \mathbb{C} \,:\, |z| \leq 1\}$ and $\mathbb{D}_\infty = \{z \in \mathbb{C} \,:\, |z| \geq 1\} \cup \{\infty\}$ and the gluing map $g\colon M \times \partial\mathbb{D}_0 \to M \times \partial\mathbb{D}_\infty$ is given by

$$g(x, e^{i\theta}) = (e^{-i\theta} \cdot x, e^{i\theta}).$$

This construction can be generalized to a loop in the group of Hamiltonian diffeomorphims and yields a Hamiltonian bundle $E \to \mathbb{P}^1$ in general. One can equip a symplectic form ω_E on the total space E of the Hamiltonian bundle such that ω_E restricts to the symplectic form ω_M on each fibre [25].

There exists a unique $\mathbb{C}^\times$-fixed component $F_{\max} \subset M^{\mathbb{C}^\times}$ such that the normal bundle of $F_{\max}$ has only negative $\mathbb{C}^\times$-weights. For a Hamiltonian function H generating the S^1-action, $F_{\max}$ is the locus where H takes the maximum value. Each fixed point $x \in M^{\mathbb{C}^\times}$ defines a section σ_x of E. We denote by σ_0 the section associated to a fixed point in $F_{\max}$. We call it a *maximal section*. This defines a splitting[1]

$$H_2(E;\mathbb{Z}) \cong \mathbb{Z}[\sigma_0] \oplus H_2(M;\mathbb{Z}). \tag{2}$$

Let $\mathrm{NE}(M) \subset H_2(M,\mathbb{R})$ denote the Mori cone, that is the cone generated by effective curves and set $\mathrm{NE}(M)_{\mathbb{Z}} := \{d \in H_2(M;\mathbb{Z}) : d \in \mathrm{NE}(M)\}$. We introduce $\mathrm{NE}(E)$ and $\mathrm{NE}(E)_{\mathbb{Z}}$ similarly.

Lemma 2.6 ([19, Lemma 2.2])**.** $\mathrm{NE}(E)_{\mathbb{Z}} = \mathbb{Z}_{\geq 0}[\sigma_0] + \mathrm{NE}(M)_{\mathbb{Z}}$.

Let $H_2^{\mathrm{sec}}(E;\mathbb{Z})$ denote the affine subspace of $H_2(E;\mathbb{Z})$ which consists of section classes, i.e. the classes that project to the positive generator of $H_2(\mathbb{P}^1;\mathbb{Z})$. We set $\mathrm{NE}(E)_{\mathbb{Z}}^{\mathrm{sec}} := \mathrm{NE}(E)_{\mathbb{Z}} \cap H_2^{\mathrm{sec}}(E;\mathbb{Z})$. The above lemma shows that

$$\mathrm{NE}(E)_{\mathbb{Z}}^{\mathrm{sec}} = [\sigma_0] + \mathrm{NE}(M)_{\mathbb{Z}}. \tag{3}$$

For $d \in \mathrm{NE}(X)_{\mathbb{Z}}$ and $\sigma \in \mathrm{NE}(E)_{\mathbb{Z}}$, we denote by q^d and q^σ the corresponding elements in the group ring $\mathbb{Q}[\mathrm{NE}(X)_{\mathbb{Z}}]$ and $\mathbb{Q}[\mathrm{NE}(E)_{\mathbb{Z}}]$ respectively. We write:

$$q^\sigma = q_0^k q^d \qquad \text{when} \quad \sigma = k[\sigma_0] + d$$

where $q_0 = q^{\sigma_0}$ is the variable corresponding to the maximal section σ_0. For $\sigma \in \mathrm{NE}(E)_{\mathbb{Z}}^{\mathrm{sec}}$, let $\mathcal{M}_{\mathrm{S}}(\sigma)$ denote the moduli space of stable maps from genus-zero closed nodal Riemann surfaces to E in the class σ with one marked point whose image lies in a fixed fibre $M \subset E$. We can write

$$\mathcal{M}_{\mathrm{S}}(\sigma) = \mathcal{M}_1(\sigma) \times_E M$$

using the usual moduli space $\mathcal{M}_1(\sigma)$ of genus-zero one-pointed stable maps to E in the class σ. Since $\mathcal{M}_1(\sigma)$ has a Kuranishi structure (without boundary) of virtual real dimension $2n + 2\langle c_1(E), \sigma\rangle - 2$ (with $n := \dim_{\mathbb{C}} M$) and we may assume that the evaluation map $\mathcal{M}_1(\sigma) \to E$ is weakly submersive, the fibre product $\mathcal{M}_{\mathrm{S}}(\sigma)$ is equipped with the induced Kuranishi structure of virtual dimension:

$$\mathrm{vir.\,dim}_{\mathbb{R}}\, \mathcal{M}_{\mathrm{S}}(\sigma) = 2n + 2\left\langle c_1^{\mathrm{vert}}(E), \sigma\right\rangle. \tag{4}$$

Here $c_1^{\mathrm{vert}}(E)$ denotes the 1st Chern class of the vertical tangent bundle $T_{\mathrm{vert}}E$,

$$T_{\mathrm{vert}}E := \mathrm{Ker}(d\pi \colon TE \to \pi^* T\mathbb{P}^1)$$

with $\pi \colon E \to \mathbb{P}^1$ the natural projection. (Note that $\langle c_1(E), \sigma\rangle = \langle c_1^{\mathrm{vert}}(E), \sigma\rangle + 2$.) Let $\mathrm{ev}\colon \mathcal{M}_{\mathrm{S}}(\sigma) \to M$ be the evaluation map and let $[\mathcal{M}_{\mathrm{S}}(\sigma)]^{\mathrm{vir}}$ be the virtual fundamental cycle of $\mathcal{M}_{\mathrm{S}}(\sigma)$.

[1] The section σ_0 gives a splitting of the Serre spectral sequence. In general one has a non-canonical splitting $H^*(E;\mathbb{Q}) \cong H^*(M;\mathbb{Q}) \otimes H^*(\mathbb{P}^1;\mathbb{Q})$ for any Hamiltonian bundle $E \to \mathbb{P}^1$ [23].

Definition 2.7. The *Seidel element* associated to the $\mathbb{C}^\times$-action on M is the class

$$S := \sum_{\sigma \in \mathrm{NE}(E)_{\mathbb{Z}}^{\mathrm{sec}}} \mathrm{PD}\left(\mathrm{ev}_*[\mathcal{M}_S(\sigma)]^{\mathrm{vir}}\right) q^\sigma \tag{5}$$

in $H^*(M;\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{NE}(E)_{\mathbb{Z}}]\!]$. Here PD stands for the Poincaré duality isomorphism. By (3), we can factorize S as $S = q_0 \widetilde{S}$ with $\widetilde{S}$ in the small quantum cohomology ring

$$QH(M) := H(X;\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{NE}(M)_{\mathbb{Z}}]\!]$$

and $q_0 := q^{\sigma_0}$ as above. Then $\widetilde{S}$ is an invertible element of $QH(M)[q^{-d} : d \in \mathrm{NE}(M)_{\mathbb{Z}}]$ with respect to the quantum product [25, 22, 23].

Remark 2.8. Using genus zero one-point Gromov-Witten invariants for E, we can write

$$S = \sum_{\sigma \in \mathrm{NE}(E)_{\mathbb{Z}}^{\mathrm{sec}}} \sum_i \langle \iota_* \phi_i \rangle_{0,1,\sigma}^E \, \phi^i q^\sigma$$

where $\{\phi_i\}$ is a basis of $H^*(M;\mathbb{Q})$, $\{\phi^i\}$ is the dual basis with respect to the Poincaré pairing and $\iota\colon M \to E$ is the inclusion of a fibre. (We followed the standard notation of Gromov-Witten invariants as in [10].)

Remark 2.9. For a general symplectic manifold M, we use the Novikov ring Λ

$$\Lambda := \left\{ \textstyle\sum_{d \in H_2(M;\mathbb{Z})} c_d q^d \ : \ c_d \in \mathbb{Q}, \ \sharp\{d \ : \ c_d \neq 0, \ \langle \omega, d\rangle \le E\} < \infty \text{ for all } E \in \mathbb{R} \right\}.$$

instead of $\mathbb{Q}[\![\mathrm{NE}(M)_{\mathbb{Z}}]\!]$. The Seidel elements associated to loops in $\mathrm{Ham}(M,\omega)$ define a group homomorphism [25, 22, 23]:

$$\pi_1(\mathrm{Ham}(M,\omega)) \to QH(M)_\Lambda^\times \Big/ \left\{ q^d \ : \ d \in H_2(M;\mathbb{Z}) \right\}$$

which is called the *Seidel representation*, where $QH(M)_\Lambda = H^*(M;\mathbb{Q}) \otimes \Lambda$ denotes the quantum cohomology ring over Λ.

3. Degeneration Formula

Let M be a smooth projective variety equipped with a $\mathbb{C}^\times$-action. We take an S^1-invariant Kähler form ω on M. Let L be a Lagrangian submanifold of M which is preserved by $S^1 \subset \mathbb{C}^\times$, i.e. $\lambda L \subset L$ for $\lambda \in S^1$. Instead of counting holomorphic discs in (M, L), we shall consider the problem of counting holomorphic *disc sections* of the bundle $M \times \mathbb{D} \to \mathbb{D}$ with boundary in $L \times S^1$. Then we degenerate the target $M \times \mathbb{D}$ into the union of the associated bundle E and $M \times \mathbb{D}$. From this we expect a certain relationship between Seidel elements and disc counting invariants. We assume that M is a smooth projective variety with a $\mathbb{C}^\times$-action for simplicity, but the degeneration formula in this section makes sense for a symplectic manifold with a Hamiltonian circle action (or a loop in the group of Hamiltonian diffeomorphisms) in general.

3.1. **Degeneration of $M\times\mathbb{D}$.** Let $\mathbb{D}$ denote the unit disc $\{z\in\mathbb{C} : |z|\le 1\}$. A degeneration of the disc $\mathbb{D}$ into the union $\mathbb{D}\cup\mathbb{P}^1$ is given by the blowup $\mathrm{Bl}_{(0,0)}(\mathbb{D}\times\mathbb{C})$ of $\mathbb{D}\times\mathbb{C}$ at the origin. The projection $\pi\colon \mathrm{Bl}_{(0,0)}(\mathbb{D}\times\mathbb{C})\to\mathbb{C}$ satisfies $\pi^{-1}(t)\cong\mathbb{D}$ for $t\neq 0$ and $\pi^{-1}(0)\cong\mathbb{D}\cup\mathbb{P}^1$. Explicitly:

$$\mathrm{Bl}_{(0,0)}(\mathbb{D}\times\mathbb{C})=\left\{(z,t,[\alpha,\beta])\in\mathbb{D}\times\mathbb{C}\times\mathbb{P}^1 : z\beta-t\alpha=0\right\}.$$

An M-bundle $\mathcal{E}$ over $\mathrm{Bl}_{(0,0)}(\mathbb{D}\times\mathbb{C})$ is defined as follows.

$$\mathcal{E}:=\left\{(x,z,t,(\alpha,\beta))\in M\times\mathbb{D}\times\mathbb{C}\times(\mathbb{C}^2\setminus\{0\}) : z\beta-t\alpha=0\right\}/\mathbb{C}^\times$$

where $\mathbb{C}^\times$ acts as $(x,z,t,(\alpha,\beta))\mapsto(\lambda x,z,t,(\lambda\alpha,\lambda\beta))$. We have a natural projection $\pi\colon\mathcal{E}\to\mathbb{C}$. One can see that

$$\mathcal{E}_t=\pi^{-1}(t)=\begin{cases}M\times\mathbb{D} & \text{if } t\neq 0;\\ E\cup_M(M\times\mathbb{D}) & \text{if } t=0\end{cases}\tag{6}$$

where E is the associated bundle (Definition 2.4) of the $\mathbb{C}^\times$-action on M. One can also construct $\mathcal{E}$ as a symplectic quotient:

$$\mathcal{E}=\left\{(x,z,t,(\alpha,\beta)) : z\beta-t\alpha=0,\ H(x)+|\alpha|^2+|\beta|^2=c\right\}/S^1$$

where $H\colon M\to\mathbb{R}$ is the moment map of the S^1-action and $c>\max_{x\in M}H(x)$ is a real number. We can equip $\mathcal{E}$ with a symplectic structure. The boundary $\partial\mathcal{E}_t$ can be identified with $M\times S^1$ via the map:

$$M\times S^1\ni(x,z)\mapsto[x,z,t,(z,t)]\in\partial\mathcal{E}_t.\tag{7}$$

Via this identification, $\mathcal{E}_t$ contains a Lagrangian submanifold $\widehat{L}_t:=L\times S^1$ in the boundary $M\times S^1\cong\partial\mathcal{E}_t$.

We can also close $\mathcal{E}_t$ by attaching $M\times\mathbb{D}$ to the boundary for each t and get a degenerating family $\overline{\mathcal{E}}$ of closed manifolds. More explicitly, we define:

$$\overline{\mathcal{E}}=\left\{(x,(z,w),t,(\alpha,\beta))\in M\times(\mathbb{C}^2\setminus\{0\})\times\mathbb{C}\times(\mathbb{C}^2\setminus\{0\}) : t\alpha w=z\beta\right\}/\mathbb{C}^\times\times\mathbb{C}^\times$$

where $\mathbb{C}^\times\times\mathbb{C}^\times$ acts as

$$(x,(z,w),t,(\alpha,\beta))\mapsto(\lambda_1^{-1}\lambda_2x,(\lambda_1z,\lambda_1w),t,(\lambda_2\alpha,\lambda_2\beta)).$$

This is an M-bundle over

$$\mathrm{Bl}_{(0,0)}(\mathbb{P}^1\times\mathbb{C})=\left\{(z,w),t,(\alpha,\beta)\in\mathbb{P}^1\times\mathbb{C}\times\mathbb{P}^1 : t\alpha w=z\beta\right\}.$$

With respect to the projection $\pi\colon\overline{\mathcal{E}}\to\mathbb{C}$ to the t-plane, we have

$$\overline{\mathcal{E}}_t=\pi^{-1}(t)=\begin{cases}M\times\mathbb{P}^1 & \text{if } t\neq 0;\\ E\cup_M E' & \text{if } t=0.\end{cases}$$

where E' is the associated bundle of the $\mathbb{C}^\times$-action on M *inverse* to the original one. Note that $\mathcal{E}$ is contained in $\overline{\mathcal{E}}$ as the locus $\{w=1,\ |z|\le 1\}$ and $\overline{\mathcal{E}}=\mathcal{E}\cup_{M\times S^1\times\mathbb{C}}(M\times\mathbb{D}^2\times\mathbb{C})$. We can also equip $\overline{\mathcal{E}}$ with a symplectic structure by describing it as a symplectic quotient in a similar manner.

A topological description is given as follows. We start from a trivial M-bundle $M\times\mathbb{P}^1$ over $\mathbb{P}^1$. We cut $\mathbb{P}^1$ into 3 pieces: $\mathbb{P}^1=\mathbb{D}_0\cup A\cup\mathbb{D}_\infty$, where $\mathbb{D}_0=\{|z|\le 1/2\}$, $A=\{1/2\le$

$|z| \le 2\}$ and $\mathbb{D}_\infty = \{|z| \ge 2\} \cup \{\infty\}$. One can twist the clutching function along $\partial\mathbb{D}_0$ and $\partial\mathbb{D}_\infty$ by the given S^1-action on M; namely

$$M \times \mathbb{P}^1 = (M \times \mathbb{D}_0) \cup_{g_1} (M \times A) \cup_{g_2} (M \times \mathbb{D}_\infty) \tag{8}$$

where the clutching functions g_1, g_2 are given respectively by

$$g_1\colon M \times \partial\mathbb{D}_0 \ni (x, \tfrac{1}{2}e^{i\theta}) \longmapsto (e^{-i\theta}x, \tfrac{1}{2}e^{i\theta}) \in M \times \partial_0 A$$
$$g_2\colon M \times \partial_\infty A \ni (x, 2e^{i\theta}) \longmapsto (e^{i\theta}x, 2e^{i\theta}) \in M \times \partial\mathbb{D}_\infty$$

where we set $\partial A = \partial_0 A \cup \partial_\infty A$. Collapsing $M \times S^1 \subset M \times A$ down to M, we get the singular central fibre $E \cup_M E'$. In fact, for $|t| < 1$, one can decompose $\overline{\mathcal{E}}_t$ as

$$\begin{aligned}\overline{\mathcal{E}}_t =& \{[x,(tz,1),t,(z,1)] \,:\, |z| \le 1\} \\ &\cup \{[x,(z,1),t,(1,\beta)] \,:\, t = \beta z,\ |\beta| \le 1,\ |z| \le 1\} \\ &\cup \{[x,(1,w),t,(1,wt)] \,:\, |w| \le 1\}\,.\end{aligned}$$

This corresponds to the decomposition (8) of $M \times \mathbb{P}^1$ above.

Remark 3.1. We shall consider stable holomorphic discs in $(\overline{\mathcal{E}}_t, \widehat{L}_t)$ which project onto the holomorphic disc $(\mathbb{D}^2, S^1) \subset (\mathbb{P}^1, S^1)$. Such stable holomorphic discs are entirely contained in the half-space $\mathcal{E}_t$ of $\overline{\mathcal{E}}_t$, so the choice of "closing" of $\mathcal{E}_t$ is not relevant.

Remark 3.2. We can perform a similar construction for a general symplectic manifold (M, ω) equipped with a Lagrangian submanifold L and a loop $\{\phi_\theta\}_{\theta\in[0,2\pi]}$ in the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms such that $\phi_\theta(L) = L$ for all θ. We can twist the clutching function of the trivial M-bundle $M \times \mathbb{P}^1$ as in (8) where g_1, g_2 there are replaced with

$$g_1(x, \tfrac{1}{2}e^{i\theta}) = (\phi_{-\theta}(x), \tfrac{1}{2}e^{i\theta}), \quad g_2(x, 2e^{i\theta}) = (\phi_\theta(x), 2e^{i\theta}).$$

Then we can degenerate the annulus A into the union of two discs (in a one-parameter family) in the middle part $M \times A$. In the degeneration family, we have a family of Lagrangian submanifolds $L \times S^1$ lying in the boundary of $M \times \mathbb{D}_0 \cup_{g_1} M \times A$.

3.2. **Relative homology classes of degenerating discs.** We write $\mathcal{L} = \bigcup_{t\in\mathbb{C}} \widehat{L}_t$. The total space $(\mathcal{E}, \mathcal{L})$ of the family has a deformation retraction to the central fibre $(\mathcal{E}_0, \widehat{L}_0)$. This gives a retraction map for $t \ne 0$:

$$r\colon H_2(\mathcal{E}_t, \widehat{L}_t) \longrightarrow H_2(\mathcal{E}, \mathcal{L}) \cong H_2(\mathcal{E}_0, \widehat{L}_0).$$

Let $\pi\colon \mathcal{E} \to \mathrm{Bl}_{(0,0)}(\mathbb{D}\times\mathbb{C})$ denote the natural projection. We have the following commutative diagram:

$$\begin{tikzcd} H_2(\mathcal{E}_t, \widehat{L}_t) \arrow[r, "r"] \arrow[d, "\pi_*"'] & H_2(\mathcal{E}_0, \widehat{L}_0) \arrow[d, "\pi_*"'] \\ H_2(\mathbb{D}, S^1) \arrow[r, "r"] & H_2(\mathbb{P}^1 \cup \mathbb{D}, S^1) \end{tikzcd}$$

Under the natural identifications $H_2(\mathbb{D}, S^1; \mathbb{Z}) \cong \mathbb{Z}$ and $H_2(\mathbb{P}^1 \cup \mathbb{D}, S^1; \mathbb{Z}) \cong H_2(\mathbb{P}^1; \mathbb{Z}) \oplus H_2(\mathbb{D}, S^1; \mathbb{Z}) \cong \mathbb{Z}^2$, the bottom arrow is given by $n \mapsto (n, n)$. We are interested in *section classes* lying in the following groups:

$$H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) = \pi_*^{-1}(1), \text{ for } t \ne 0, \text{ and } H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0) = \pi_*^{-1}(1,1).$$

There is an induced retraction map $r\colon H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) \to H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0)$ for $t \neq 0$.

Lemma 3.3. *Assume that M is simply connected and L is connected. Then we have*

$$\begin{aligned} &H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) \cong H_2(M, L) \quad \textit{for } t \neq 0 \\ &H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0) \cong H_2^{\mathrm{sec}}(E) \times_{H_2(M)} H_2(M, L) \end{aligned} \tag{9}$$

Proof. Recall that $(\mathcal{E}_t, \widehat{L}_t) \cong (M \times \mathbb{D}, L \times S^1)$ for $t \neq 0$. We show the isomorphism:

$$(p_{1*}, p_{2*})\colon H_2(M \times \mathbb{D}, L \times S^1) \cong H_2(M, L) \times H_2(\mathbb{D}, S^1)$$

where p_1, p_2 are natural projections. Because we have sections $i_1, i_2\colon (M, L) \to (M \times \mathbb{D}, L \times S^1)$ such that $p_1 \circ i_1 = \mathrm{id}$, $p_2 \circ i_2 = \mathrm{id}$, $p_2 \circ i_1 = \mathrm{const}$ and $p_1 \circ i_2 = \mathrm{const}$, the map (p_{1*}, p_{2*}) is surjective. To show that it is injective, we use the commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & H_2(\mathbb{D}, S^1) \arrow[r] & H_1(S^1) \\ H_2(L \times S^1) \arrow[r] \arrow[u] \arrow[d, "\mathrm{epi}"'] & H_2(M \times \mathbb{D}) \arrow[r] \arrow[u] \arrow[d, "\cong"'] & H_2(M \times \mathbb{D}, L \times S^1) \arrow[r] \arrow[u, "p_{2*}"] \arrow[d, "p_{1*}"'] & H_1(L \times S^1) \arrow[u] \arrow[d] \\ H_2(L) \arrow[r] & H_2(M) \arrow[r] & H_2(M, L) \arrow[r] & H_1(L). \end{tikzcd}$$

Here all the horizontal sequences are exact. The injectivity of (p_{1*}, p_{2*}) follows from the diagram chasing and $H_1(L \times S^1) \cong H_1(L) \oplus H_1(S^1)$ (here we use the condition that L is connected). Then $H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) \cong H_2(M, L)$ for $t \neq 0$ follows.

The Mayer-Vietoris exact sequence for $\mathcal{E}_0 = E \cup_M (M \times \mathbb{D})$ gives

$$H_2(M) \longrightarrow H_2(E) \oplus H_2(M \times \mathbb{D}, L \times S^1) \longrightarrow H_2(\mathcal{E}_0, \widehat{L}_0) \longrightarrow 0$$

Here we used $H_1(M) = 0$. The formula for $H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0)$ follows. $\square$

Henceforth we assume that L *is connected and* M *is simply-connected*.

Remark 3.4. The natural map $H_2(\mathcal{E}_t, \widehat{L}_t) \to H_2(\overline{\mathcal{E}}_t, \widehat{L}_t)$ is injective because the composition:

$$H_2(M \times \mathbb{D}, L \times S^1) \to H_2(M \times \mathbb{P}^1, L \times S^1) \xrightarrow{(p_{1*}, p_{2*})} H_2(M, L) \oplus H_2(\mathbb{P}^1, S^1)$$

is injective.

Notation 3.5. We denote by

$$\begin{aligned} \hat{\beta} &\in H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) \cong H_2^{\mathrm{sec}}(M \times \mathbb{D}, L \times S^1) \quad (t \neq 0) \\ \sigma + \hat{\beta} &\in H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0) \end{aligned}$$

the homology classes corresponding to $\beta \in H_2(M, L)$ and to $[\sigma, \beta] \in H_2^{\mathrm{sec}}(E) \times_{H_2(M)} H_2(M, L)$ respectively, under the isomorphism (9) in Lemma 3.3.

Let $u\colon \mathbb{D} \to M$ be a disc such that $u(e^{i\theta}) = e^{i\theta} \cdot u(1)$, namely, u is a disc contracting an S^1-orbit in M. This defines a section $\sigma(u)$ of the associated bundle $E \to \mathbb{P}^1$:

$$\begin{aligned} \sigma(u)|_{\mathbb{D}_0} &: \mathbb{D}_0 \to E|_{\mathbb{D}_0} \cong M \times \mathbb{D}_0, \quad z \mapsto (z, u(1)) \\ \sigma(u)|_{\mathbb{D}_\infty} &: \mathbb{D}_\infty \to E|_{\mathbb{D}_\infty} \cong M \times \mathbb{D}_\infty, \quad z \mapsto (z, u(z^{-1})) \end{aligned}$$

where $\mathbb{D}_0 = \{z \in \mathbb{C} : |z| \le 1\}$ and $\mathbb{D}_\infty = \{z \in \mathbb{C} : |z| \ge 1\} \cup \{\infty\}$; here we used the gluing construction of E in Remark 2.5.

Recall the maximal section class σ_0 of E in §2.2. We introduce a similar *maximal disc class* $\alpha_0 \in H_2(M, L)$ as follows. Take a path $\gamma\colon [0,1] \to M$ such that $\gamma(0) \in F_{\max}$ and $\gamma(1) \in L$, where $F_{\max}$ is the maximal fixed component. We define α_0 to be the class represented by the disc $\mathbb{D} \ni re^{i\theta} \mapsto e^{-i\theta} \cdot \gamma(r) \in M$. The homotopy class here is independent of the choice of a path γ because M is simply-connected and L is connected. The boundary of α_0 is an inverse S^1-orbit on L.

Proposition 3.6. *The retraction map $r\colon H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) \to H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0)$ (for $t \neq 0$) of section classes is an isomorphism. It is given by (under Notation 3.5)*

$$r(\hat{\beta}) = \sigma(u) - \hat{u} + \hat{\beta} = \sigma_0 + \hat{\alpha}_0 + \hat{\beta} \qquad \textit{for } \beta \in H_2(M, L)$$

where $u\colon \mathbb{D} \to M$ is an arbitrary disc whose boundary is an S^1-orbit in L and σ_0, α_0 are the maximal classes. In particular we have the commutative diagram

$$\begin{array}{ccc} H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) & \xrightarrow[r]{\cong} & H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0) \\ {\scriptstyle\partial}\big\downarrow & & {\scriptstyle\partial}\big\downarrow \\ H_1(L) & \xrightarrow[-\lambda]{\cong} & H_1(L) \end{array}$$

where the bottom map is the subtraction of the class $\lambda = [\partial u]$ of an S^1-orbit on L.

Proof. Consider a constant section $s_{\mathrm{triv}}(z) = (x, z)$ of $M \times \mathbb{D} \cong \mathcal{E}_t$ with $x \in L$. By the topological description of the degeneration given in §3.1, we see that s_{triv} can degenerate to the union:

$$\sigma(u) \cup \tilde{u}\colon \mathbb{P}^1 \cup \mathbb{D} \to E \cup_M (M \times \mathbb{D})$$

where $u\colon \mathbb{D} \to M$ is a disc contracting the S^1-orbit $e^{i\theta}x$ on L and $\tilde{u}\colon \mathbb{D} \to M \times \mathbb{D}$ is given by $z \mapsto (u(\overline{z}), z)$. This shows that $r([s_{\mathrm{triv}}]) = \sigma(u) - \hat{u}$. Since the retraction map is a homomorphism of $H_2(M, L)$-modules, we have $r(\hat{\beta}) = \sigma(u) - \hat{u} + \hat{\beta}$ in general. When u is a disc of the form: $\mathbb{D} \ni re^{i\theta} \mapsto e^{i\theta} \cdot \gamma(r) \in M$, where $\gamma\colon [0,1] \to M$ is a path such that $\gamma(0) \in F_{\max}$ and $\gamma(1) \in L$, $\sigma(u)$ is homotopic to the maximal section σ_0 and $[u] = -\alpha_0$. This shows the formula $r(\hat{\beta}) = \sigma_0 + \hat{\alpha}_0 + \hat{\beta}$. It is easy to check that r is an isomorphism between section classes. □

Remark 3.7. The latter statement is a consequence of the difference of trivializations of $\partial\mathcal{E}_t$ ($t \neq 0$) and $\partial\mathcal{E}_0$. Recall that we have a trivialization $\partial\mathcal{E}_t \cong M \times S^1$ in (7) depending smoothly on $t \in \mathbb{C}$. For $t \neq 0$, this trivialization is induced from the isomorphism $\mathcal{E}_t \cong M \times \mathbb{D}$ in (6); however for $t = 0$, this trivialization differs by the S^1-action from the one induced by the isomorphism $\mathcal{E}_0 \cong E \cup_M (M \times \mathbb{D})$ in (6).

Lemma 3.8 (Maslov index and vertical Chern number)**.** *Let $u\colon \mathbb{D} \to M$ be a disc with boundary an S^1-orbit on L, i.e. $u(e^{i\theta}) = e^{i\theta} \cdot u(1)$ and $u(1) \in L$. Then u defines a class in $\pi_2(M, L)$ and we have $\mu(u) = 2\,\langle c_1^{\mathrm{vert}}(E), [\sigma(u)]\rangle$.*

Proof. We recall the definition of Maslov index of a disc $u\colon (\mathbb{D}, S^1) \to (M, L)$. We set $\gamma = u|_{\partial\mathbb{D}}$. Note that $u^*TM|_{S^1}$ is a complexfication of the subbundle γ^*TL. Thus $\det(u^*TM)|_{S^1}$

is a complexification of the real line bundle $\det_{\mathbb{R}}(\gamma^* TL)$. On the other hand $\det_{\mathbb{R}}(\gamma^* TL)^{\otimes 2}$ has a canonical orientation. Take a positive (nowhere vanishing) section s_0 of $\det_{\mathbb{R}}(\gamma^* TL)^{\otimes 2}$. The Maslov index of u is the signed count of zeros of a transverse section s of $\det(u^*TM)^{\otimes 2}$ such that $s|_{\partial\mathbb{D}} = s_0$.

When $u|_{\partial\mathbb{D}}$ is an S^1-orbit of L, we can take s_0 above to be S^1-equivariant. A transverse section s of $\det(u^*TM)^{\otimes 2}$ with $s|_{\partial\mathbb{D}} = s_0$ defines a section $t \in \det(\sigma(u)^* T_{\mathrm{vert}} E)^{\otimes 2}$ by

$$t|_{\mathbb{D}_0}(z) = s_0(1), \quad t|_{\mathbb{D}_\infty}(z) = s(z^{-1}).$$

Then the numbers of zeros of t and s coincide. The lemma follows. $\square$

Proposition 3.6 and Lemma 3.8 show the following corollary:

Corollary 3.9. *Let $r\colon H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t) \to H_2^{\mathrm{sec}}(\mathcal{E}_0, \widehat{L}_0)$ be the retraction map for $t \neq 0$. Suppose that $r(\hat{\beta}) = \sigma + \hat{\alpha}$ with $\alpha, \beta \in H_2(M, L)$. Then $\mu(\beta) = 2\,\langle c_1^{\mathrm{vert}}(E), \sigma\rangle + \mu(\alpha)$.*

Remark 3.10. We have $\mu(\hat{\beta}) = \mu(\beta) + 2$ for $\beta \in H_2(M, L)$.

3.2.1. *Example.* We give an example of degenerating holomorphic discs. Consider a family of (constant) holomorphic disc sections $u_t\colon (\mathbb{D}, S^1) \to (\mathcal{E}_t, \widehat{L}_t)$ given by

$$u_t(z) = [x_0, z, t, (z, t)]$$

for some $x_0 \in L$. For a fixed non-zero $z \in \mathbb{D}$, we have

$$\varphi(z) := \lim_{t\to 0} u_t(z) = [x_0, z, 0, (z, 0)]$$

This can be completed to a holomorphic disc section $\varphi\colon \mathbb{D} \to M \times \mathbb{D} \subset \mathcal{E}_0$. Note that the limit

$$\lim_{z\to 0} \varphi(z) = [x_1, z, 0, (1, 0)] \qquad \text{where} \quad x_1 := \lim_{z\to 0} z^{-1} x_0 \in M,$$

exists by the completeness of M. On the other hand, we can see a bubbling off holomorphic sphere at $z = 0$ by the usual rescaling:

$$\psi(z) := \lim_{t\to 0} u_t(tz) = \lim_{t\to 0} [t^{-1} x_0, tz, t, (z, 1)] = [x_1, 0, 0, (z, 1)].$$

This defines a holomorphic section $\psi\colon \mathbb{P}^1 \to E \subset \mathcal{E}_0$ associated to the $\mathbb{C}^\times$-fixed point $x_1 \in M$. Note that $\psi(\infty) = \varphi(0)$ and $\partial\varphi$ is an inverse S^1-orbit on L.

3.3. **Degeneration formula.** In what follows, we propose a conjectural degeneration formula and discuss its consequences. As before, M denotes a smooth projective variety equipped with a $\mathbb{C}^\times$-action and an S^1-invariant Kähler form ω; L is a Lagrangian submanifold which is preserved by the S^1-action. We assume that M is simply-connected and L is connected. Moreover we assume that L is oriented and relatively spin and we fix a relative spin structure [12, Definition 8.1.2].

Take $\beta \in H_2(M, L)$. We consider the moduli space $\mathcal{M}_1(\hat{\beta})$ of stable holomorphic maps from genus zero bordered Riemann surface $(\Sigma, \partial\Sigma)$ to $(\overline{\mathcal{E}}_t, \widehat{L}_t) \cong (M \times \mathbb{P}^1, L \times S^1)$ with one boundary marked point and in the class $\hat{\beta} \in H_2^{\mathrm{sec}}(\mathcal{E}_t, \widehat{L}_t)$ (where $t \neq 0$; see Notation 3.5). Such stable maps project onto the disc $(\mathbb{D}, S^1) \subset (\mathbb{P}^1, S^1)$ on the base and so are contained in $\mathcal{E}_t$ (see Remarks 3.1 and 3.4). The virtual dimension of $\mathcal{M}_1(\hat{\beta})$ is $n + 1 + \mu(\hat{\beta}) - 2 =$

$n + 1 + \mu(\beta)$ with $n := \dim_{\mathbb{C}} M$. The corresponding moduli space at $t = 0$ should be described as the fibre product:

$$\bigcup_{r(\hat{\beta})=\sigma+\hat{\alpha}} \mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$$

where $\mathcal{M}_{\mathrm{S}}(\sigma)$ is the moduli space of holomorphic sections of E appearing in §2.2 and $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ is the moduli space of stable holomorphic maps from genus zero bordered Riemann surfaces to $(M \times \mathbb{P}^1, L \times S^1)$ in the class $\hat{\alpha} \in H_2^{\mathrm{sec}}(M \times \mathbb{D}, L \times S^1)$ with one boundary marked point and one interior marked point such that the image of the interior marked point lies in $M \times \{0\}$. The superscript "rel" (which means "relative") signifies the last condition. The fibre product above is taken with respect to the interior evaluation maps. One can write:

$$\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha}) = \mathcal{M}_{1,1}(\hat{\alpha}) \times_{M\times\mathbb{P}^1} (M \times \{0\}) \tag{10}$$

using the moduli space $\mathcal{M}_{1,1}(\hat{\alpha})$ of bordered stable maps to $(M \times \mathbb{P}^1, L \times S^1)$ of class $\hat{\alpha}$ with one boundary marking and one interior marking. Then a Kuranishi structure on $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ is induced from the Kuranishi structure on $\mathcal{M}_{1,1}(\hat{\alpha})$ (as defined in [12, §7.1]) via this presentation. The virtual dimension is

$$\mathrm{vir.}\dim \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha}) = n + 1 + \mu(\alpha).$$

We write $\mathrm{ev}^{(\mathrm{i})} \colon \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha}) \to M$ for the interior evaluation map and $\mathrm{ev}^{(\mathrm{b})} \colon \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha}) \to L \times S^1$ for the boundary evaluation map.

When the virtual fundamental *chains* on the moduli spaces $\mathcal{M}_1(\hat{\beta})$ and $\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ happen to be *cycles*, we expect the following degeneration formula:

$$\varphi_* \operatorname{ev}_*[\mathcal{M}_1(\hat{\beta})]^{\mathrm{vir}} = \sum_{r(\hat{\beta})=\sigma+\hat{\alpha}} \operatorname{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}} \tag{11}$$

in $H_*(L \times S^1)$. Here ev on the both-hand sides denotes the evaluation map at the boundary markings taking values in $\widehat{L}_t \cong L \times S^1$ and $\varphi \colon L \times S^1 \to L \times S^1$ is the map $(x, e^{i\theta}) \mapsto (e^{-i\theta} \cdot x, e^{i\theta})$ which corresponds to the difference of boundary trivializations (see Remark 3.7). We will study below when the both-hand sides of (11) make sense as cycles; then will calculate them in terms of Seidel elements and open Gromov-Witten invariants.

3.3.1. *The left-hand side of* (11). When $\beta = 0$, $\mathcal{M}_1(\hat{\beta})$ consists of constant disc sections and $\mathrm{ev} \colon \mathcal{M}_1(\hat{\beta}) \to L \times S^1$ is a homeomorphism. All constant disc sections are Fredholm regular. When $\beta \neq 0$, we have a natural map

$$\mathcal{M}_1(\hat{\beta}) \to \mathcal{M}_1(\beta)$$

induced by the projection $\mathcal{E}_t \to M$, where $\mathcal{M}_1(\beta)$ is the moduli space of one-pointed bordered stable maps to (M, L) in the class β. By taking the graph of a disc component, we can see that this map is surjective. Therefore, for $\beta \neq 0$, $\mathcal{M}_1(\hat{\beta})$ is non-empty if and only if $\mathcal{M}_1(\beta)$ is non-empty. Moreover, if $\mathcal{M}_1(\beta)$ is non-empty, $\mathcal{M}_1(\hat{\beta})$ has boundary since a bordered stable map of class $\hat{\beta}$ can be constructed as a union of a constant disc section and a disc of class β (which is constant in the $\mathbb{D}$-direction). Therefore, we have

Lemma 3.11. *The virtual cycle* $\mathrm{ev}_*[\mathcal{M}_1(\hat{\beta})]^{\mathrm{vir}}$ *is well-defined if* $\mathcal{M}_1(\beta) = \emptyset$. *We have*

$$\varphi_* \mathrm{ev}_*[\mathcal{M}_1(\hat{\beta})]^{\mathrm{vir}} = \begin{cases} [L \times S^1] & \text{if } \beta = 0; \\ 0 & \text{if } \beta \neq 0 \text{ and } \mathcal{M}_1(\beta) = \emptyset. \end{cases}$$

3.3.2. *The right-hand side of* (11). Take $(\sigma, \alpha) \in H_2^{\mathrm{sec}}(E) \times H_2(M, L)$ such that $\sigma + \hat{\alpha} = r(\hat{\beta})$. By Corollary 3.9 and Proposition 3.6, we have

$$\mu(\beta) = 2\left\langle c_1^{\mathrm{vert}}(E), \sigma \right\rangle + \mu(\alpha) \tag{12}$$

$$\partial\beta = \partial\alpha + \lambda \tag{13}$$

where $\lambda \in H_1(L)$ is the class of an S^1-orbit.

Suppose $\alpha = 0$. This can happen only when $\partial\beta = \lambda$ by (13). Since $\alpha = 0$, $\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$ consists of constant disc sections and $\mathrm{ev}^{(\mathrm{b})} \colon \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha}) \cong L \times S^1$. The interior evaluation $\mathrm{ev}^{(\mathrm{i})} \colon \mathcal{M}_{1,1}^{\mathrm{rel}}(\alpha) \to M$ is given by the projection $L \times S^1 \to L \subset M$. Thus

$$\begin{aligned} \mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})]^{\mathrm{vir}} &= \mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M (L \times S^1)]^{\mathrm{vir}} \\ &= (\mathcal{S}_\sigma \cap [L]) \times [S^1] \end{aligned} \tag{14}$$

where

$$\mathcal{S}_\sigma := \mathrm{PD}\left(\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma)]^{\mathrm{vir}}\right) \in H^{-\mu(\beta)}(M). \tag{15}$$

Here we used the virtual dimension formula (4) and (12).

Suppose $\alpha \neq 0$. By the same argument as in §3.3.1, $\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$ is non-empty if and only if $\mathcal{M}_1(\alpha)$ is non-empty; also $\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$ has boundary if $\mathcal{M}_1(\alpha)$ is non-empty. Assume that $\mathcal{M}_1(\alpha)$ has no boundary. This means that every stable map in $\mathcal{M}_1(\alpha)$ has only one disc component[2] (but possibly with sphere bubbles). Let us study the moduli space $\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})$ and its boundary. Since $\alpha \neq 0$, we have a map

$$\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2) \colon \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha}) \to \mathcal{M}_1(\alpha) \times S^1. \tag{16}$$

The first factor $\mathfrak{f}_1$ is given by projecting bordered stable maps to M, forgetting the interior marking and collapsing unstable components; the second factor $\mathfrak{f}_2$ is the boundary evaluation $\mathrm{ev}^{(\mathrm{b})} \colon \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha}) \to L \times S^1$ followed by the projection $L \times S^1 \to S^1$. The map $\mathfrak{f}$ can be viewed as a tautological family of stable discs over $\mathcal{M}_1(\alpha) \times S^1$. In fact we have the following result.

Lemma 3.12. *Let* $u \colon (\Sigma, \partial\Sigma) \to (M, L)$ *be a one-pointed bordered stable map of class* α *and* $x \in \partial\Sigma$ *be the boundary marking. Suppose that* Σ *has only one disc component. Then the fibre* $\mathfrak{f}^{-1}([u, \Sigma, x], z)$ *at* $([u, \Sigma, x], z) \in \mathcal{M}_1(\alpha) \times S^1$ *can be identified with the oriented real blow-up* $\widehat{\Sigma}$ *of* Σ *at* x *(see the proof below for the definition of* $\widehat{\Sigma}$*) and the interior evaluation* $\mathrm{ev}^{(\mathrm{i})}$ *on* $\mathfrak{f}^{-1}([u, \Sigma, x], z)$ *can be identified with the map* $\widehat{\Sigma} \to \Sigma \xrightarrow{u} M$.

Proof. The assumption that Σ has only one disc component was made for simplicity's sake (and this is the case we are interested in). In general, the fibre of $\mathfrak{f}$ can be identified with a smoothing of $\widehat{\Sigma}$ at the boundary singularities. See [12, Lemma 7.1.45] for a similar statement.

[2] See [12, §7.1.1] for the boundary description of the moduli spaces.

We identify a neighbourhood of $x \in \Sigma$ with the upper-half disc $\mathbb{D}_+ = \{w \in \mathbb{D} : \mathrm{Im}(w) \geq 0\}$ where x corresponds to $0 \in \mathbb{D}_+$. The oriented real blow-up $\widehat{\Sigma}$ is defined by replacing this neighbourhood with $[0,\pi] \times [0,1]$:

$$\widehat{\Sigma} = (\Sigma \setminus \{x\}) \cup_{\mathbb{D}_+ \setminus \{0\}} ([0,\pi] \times [0,1])$$

where $\mathbb{D}_+ \setminus \{0\}$ is identified with $[0,\pi] \times (0,1]$ by the map $w \mapsto (\arg(w), |w|)$. Note that $\widehat{\Sigma}$ is a real analytic manifold (with boundary and corner) equipped with a natural projection $\widehat{\Sigma} \to \Sigma$.

For a point $p \in \widehat{\Sigma}$, we shall construct a bordered stable map in the fibre $\mathfrak{f}^{-1}((u,\Sigma,x),z)$. Suppose $p \in \widehat{\Sigma} \setminus \partial\widehat{\Sigma} \cong \Sigma \setminus \partial\Sigma$. Note that Σ is a union of one disc component Σ_0 and trees of sphere bubbles. If p is in a tree of spheres bubbles, let q be the intersection point of the tree (on which p lies) and the disc Σ_0. If p is in the interior of Σ_0, set $q := p$. Take a unique holomorphic map $v\colon \Sigma_0 \to \mathbb{D}$ which sends q to $0 \in \mathbb{D}$ and $x \in \partial\Sigma_0$ to $z \in S^1$. Extend v to the whole Σ so that it is constant on each sphere component. Then we obtain a bordered stable map $\hat{u} = (u,v)\colon \Sigma \to M \times \mathbb{D}$ of class $\hat{\alpha}$ with p a new interior marked point. (If p is a node, we insert at the node a trivial sphere with an interior marking.)

Next consider the case $p \in \partial\widehat{\Sigma}$. In this case, the corresponding bordered stable map is in the boundary of $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$. See Figure 1 below. If p is not in the exceptional locus $[0,\pi]$ of $\widehat{\Sigma} \to \Sigma$, we attach a disc $\mathbb{D}$ to Σ by identifying $1 \in \partial\mathbb{D}$ with $p \in \partial\Sigma$ and define a map $\hat{u}\colon \mathbb{D} \,{}_1{\cup}_p\, \Sigma \to M \times \mathbb{D}$ by

$$\hat{u}|_{\mathbb{D}}(w) = (u(p), zw), \quad \hat{u}|_{\Sigma}(y) = (u(y), z).$$

A new interior marking is taken to be $0 \in \mathbb{D}$. If p corresponds to an interior point $\theta \in (0,\pi)$ of the exceptional locus $[0,\pi]$ of $\widehat{\Sigma} \to \Sigma$, we attach a disc $\mathbb{D}$ to Σ by identifying $1 \in \partial\mathbb{D}$ with $x \in \partial\Sigma$ and define a map $\hat{u}\colon \mathbb{D} \,{}_1{\cup}_x\, \Sigma \to M \times \mathbb{D}$ by

$$\hat{u}|_{\mathbb{D}}(w) = (u(x), e^{-2i\theta}zw), \quad \hat{u}|_{\Sigma}(y) = (u(y), e^{-2i\theta}z).$$

We put a new boundary marking at $e^{2i\theta} \in \mathbb{D}$ and a new interior marking at $0 \in \mathbb{D}$. If p is a boundary point of the exceptional locus $[0,\pi]$, say, $0 \in [0,\pi]$, we consider the domain $\mathbb{D}^{(1)} \,{}_1{\cup}_{-1}\, \mathbb{D}^{(2)} \,{}_1{\cup}_x\, \Sigma$ (subscripts signify how to identify boundary points) with a boundary marking $i \in \mathbb{D}^{(2)}$ and an interior marking $0 \in \mathbb{D}^{(1)}$ and define a map $\hat{u}\colon \mathbb{D}^{(1)} \cup \mathbb{D}^{(2)} \cup \Sigma \to M \times \mathbb{D}$ by:

$$\hat{u}|_{\mathbb{D}^{(1)}}(w) = (u(x), zw), \quad \hat{u}|_{\mathbb{D}^{(2)}}(w) = (u(x), z), \quad \hat{u}|_{\Sigma}(y) = (u(y), z).$$

When p corresponds to $\pi \in [0,\pi]$, we take $-i \in \mathbb{D}^{(2)}$ in place of $i \in \mathbb{D}^{(2)}$ as a boundary marking. One can see that the above construction defines a homeomorphism $\widehat{\Sigma} \cong \mathfrak{f}^{-1}([u,\Sigma,x],z)$. The latter statement is obvious. $\square$

From the previous lemma and its proof, we have:

Corollary 3.13. *Suppose $\partial\mathcal{M}_1(\alpha) = \emptyset$. The boundary $\partial\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ maps to L under the interior evaluation map $\mathrm{ev}^{(\mathrm{i})}\colon \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha}) \to M$.*

Corollary 3.14. *Suppose $\partial\mathcal{M}_1(\alpha) = \emptyset$ and $\mathrm{ev}(\mathcal{M}_{\mathrm{S}}(\sigma)) \cap L = \emptyset$. Then the fibre product $\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ has no boundary. In particular, the virtual fundamental cycle $\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}}$ is well-defined (see* [12, Lemma A.1.32]*).*

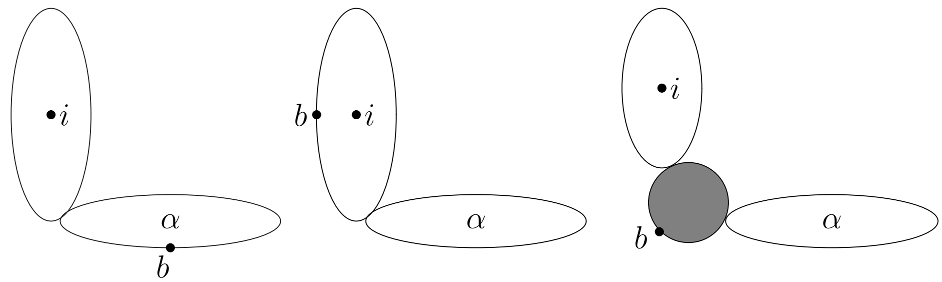

FIGURE 1. Three types of boundary points of $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$. The horizontal direction is M and the vertical direction is $\mathbb{D}$. The boundary/interior markings are denoted by b and i respectively. The shaded disc is a component where the map is constant.

We proceed to calculate the cycle $\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}}$ under the assumption of Corollary 3.14. By Corollary 3.13, taking a sufficiently small perturbation, we get a virtual fundamental *chain*

$$\mathcal{P}_\alpha := (\mathrm{ev}^{(\mathrm{i})} \times \mathrm{ev}^{(\mathrm{b})})_*[\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}}$$

whose boundary lies in $\nu(L) \times L \times S^1$, where $\nu(L) \subset M$ is an arbitrarily small tubular neighbourhood of L. In other words, $\mathcal{P}_\alpha$ defines a *relative* homology class of the pair $(M \times L \times S^1, \nu(L) \times L \times S^1)$ whose dimension is $n+1+\mu(\alpha)$ (where $n = \dim_{\mathbb{C}} M$). On the other hand, since $\mathrm{ev}(\mathcal{M}_{\mathrm{S}}(\sigma)) \cap L = \emptyset$, taking a sufficiently small perturbation again, we obtain a virtual cycle $\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma)]^{\mathrm{vir}}$ in $M \setminus \nu(L)$. By Poincaré-Lefschetz duality this defines a class

$$\widehat{\mathcal{S}}_\sigma := \mathrm{PD}\left(\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma)]^{\mathrm{vir}}\right) \in H^{\mu(\alpha)-\mu(\beta)}(M, \nu(L)). \tag{17}$$

Here we used the virtual dimension formula (4) and (12). (Note that we put "hat" to distinguish $\widehat{\mathcal{S}}_\sigma \in H^*(M, L)$ from the element $\mathcal{S}_\sigma \in H^*(M)$ appearing in (15).) The virtual cycle of the fibre product can be evaluated as the pairing of the two classes:

$$\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}} = \langle \widehat{\mathcal{S}}_\sigma, \mathcal{P}_\alpha \rangle \tag{18}$$

where $\langle \cdot, \cdot \rangle$ is the canonical pairing between relative cohomology and homology (with Künneth decomposition):

$$\begin{aligned} &H^{\mu(\alpha)-\mu(\beta)}(M, \nu(L)) \otimes H_{n+1+\mu(\alpha)}(M \times L \times S^1, \nu(L) \times L \times S^1) \\ &\quad \longrightarrow H^{\mu(\alpha)-\mu(\beta)}(M, \nu(L)) \otimes H_{\mu(\alpha)-\mu(\beta)}(M, \nu(L)) \otimes H_{n+1+\mu(\beta)}(L \times S^1) \\ &\quad \longrightarrow H_{n+1+\mu(\beta)}(L \times S^1). \end{aligned}$$

Lemma 3.15. *Suppose $\partial \mathcal{M}_1(\alpha) = \emptyset$. Then the relative homology class $\mathcal{P}_\alpha$ is given by*

$$\mathcal{P}_\alpha = \alpha \otimes \left(\mathrm{ev}_*[\mathcal{M}_1(\alpha)]^{\mathrm{vir}} \times [S^1]\right) \tag{19}$$

in $H_*(M \times L \times S^1, \nu(L) \times L \times S^1) \cong H_*(M, L) \otimes H_*(L \times S^1)$.

Proof. Consider the diagram:

$$\begin{tikzcd} \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha}) \arrow[r, "\mathfrak{f}"] \arrow[d, "\mathrm{ev}^{(\mathrm{i})}"'] & \mathcal{M}_1(\alpha) \times S^1 \arrow[r, "\mathrm{ev} \times \mathrm{id}"] & L \times S^1 \\ M & & \end{tikzcd}$$

where $\mathfrak{f}$ is given in (16). The composition of the horizontal arrows is the boundary evaluation map $\mathrm{ev}^{(\mathrm{b})}$. As we saw in Lemma 3.12, $(\mathfrak{f}, \mathrm{ev}^{(\mathrm{i})})$ can be viewed as a universal family of bordered stable maps of class α. Ignoring the issues on virtual clains, the lemma follows from this diagram. In the sequel, we compare the Kuranishi structures on $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ and $\mathcal{M}_1(\alpha)$ and see that the virtual chain of $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ can be chosen to be a "fibre bundle" over a virtual chain of $\mathcal{M}_1(\alpha) \times S^1$ with fibre the corresponding stable discs.

First, we review the construction of a Kuranishi structure on $\mathcal{M}_1(\alpha)$. We refer the reader to [12, §7.1], [15, Part 3, 4] for the details. Recall [12, Definition A1.1] that a *Kuranishi neighbourhood* of a point $\mathfrak{x}_0 \in \mathcal{M}_1(\alpha)$ is a tuple (V, E, Γ, ψ, s) where

- V is a finite dimensional manifold (possibly with boundary and corner);
- E is a finite dimensional real vector space;
- Γ is a finite group; it acts on V smoothly and effectively and on E linearly;
- s is a smooth Γ-equivariant map $V \to E$;
- ψ is a homeomorphism between $s^{-1}(0)/\Gamma$ and an open neighbourhood of $\mathfrak{x}_0$ in $\mathcal{M}_1(\alpha)$.

These data are constructed as follows. Let $(u_0 \colon \Sigma_0 \to M, x_0 \in \partial\Sigma_0)$ be a marked bordered stable map representing $\mathfrak{x}_0 \in \mathcal{M}_1(\alpha)$. The finite group Γ is given by the set of holomorphic automorphisms $\varphi\colon \Sigma_0 \to \Sigma_0$ such that $u_0 \circ \varphi = u_0$ and $\varphi(x_0) = x_0$. Since Σ_0 has only one marking, it is unstable if we forget the map u_0. We add additional interior markings $w_{0,1}, \ldots, w_{0,l}$ on Σ_0 so that Σ_0 becomes stable. We also require that the set $\{w_{0,1}, \ldots, w_{0,l}\}$ is preserved by the Γ-action [15, Definition 17.5]. Since Γ permutes the additional markings, we can regard it as a subgroup of the symmetric group $\mathfrak{S}_l$. We also take real codimension 2 submanifolds $Q_1, \ldots, Q_l$ of M such that Σ_0 intersects with Q_i transversely at $w_{0,i}$ (so u_0 is necessarily an immersion at $w_{0,i}$); moreover we require that $Q_i = Q_{\sigma(i)}$ for every permutation $\sigma \in \Gamma \subset \mathfrak{S}_l$. Let $\mathcal{M}_{1,l}$ denote the moduli space of genus-zero stable bordered Riemann surfaces with one boundary and l interior markings and let $\mathfrak{a}_0$ be the point represented by $(\Sigma_0, x_0, \{w_{0,1}, \ldots, w_{0,l}\})$. The group Γ acts on $\mathcal{M}_{1,l}$ by permutation of l interior markings and $\mathfrak{a}_0$ is fixed by Γ. Let $N \subset \mathcal{M}_{1,l}$ be a Γ-invariant small open neighbourhood[3] of $\mathfrak{a}_0$. It is equipped with a tautological family $\mathcal{R} \to N$ of stable bordered Riemann surfaces. Note that Γ also acts on $\mathcal{R}$. We take a Γ-invariant compact subset $\mathcal{K} \subset \mathcal{R}$ such that the fibre $K_0 = \Sigma_0 \cap \mathcal{K}$ at $\mathfrak{a}_0$ is the complement (in Σ_0) of small neighbourhoods of nodes of Σ_0 and that the family $\mathcal{K} \to N$ is C^∞-trivial. We choose a Γ-equivariant C^∞-trivialization $\mathcal{K} \cong K_0 \times N$ which preserves the markings. See [15, Definitions 16.2, 16.4, 16.6, 16.7]. $\mathcal{K}$ is called the "core" and its complement is called the "neck region". For a bordered Riemann surface Σ appearing as a fibre of $\mathcal{R} \to N$, the core $K = \Sigma \cap \mathcal{K}$ is identified with K_0 by the given trivialization $\mathcal{K} \cong K_0 \times N$, and thus u_0 induces a map $u_0\colon K \to M$. We consider an infinite

[3]Fukaya-Oh-Ohta-Ono [15] constructed N in two steps: first they considered a subset $\mathfrak{V} \subset \mathcal{M}_{1,l}$ consisting of deformations of $\mathfrak{a}_0$ having the same dual graph as $\mathfrak{a}_0$; then they introduced smoothing (or gluing) parameters T, θ to construct a neighbourhood N of $\mathfrak{V}$.

dimensional space $\mathcal{U}$ consisting of tuples $(u,\Sigma,x,\{w_1,\dots,w_l\})$, where $(\Sigma,x,\{w_1,\dots,w_l\})$ represents a point of N and $u\colon(\Sigma,\partial\Sigma)\to(M,L)$ is a smooth map of degree α which is sufficiently "close" to u_0 in the sense that (see [15, Definitions 17.12, 18.10])

- u is C^{10}-close to u_0 on the core $K=\Sigma\cap\mathcal{K}$;
- u is holomorphic on the neck region $\Sigma\setminus K$;
- the diameter of the image of each connected component of the neck region under u is small.

The group Γ acts on $\mathcal{U}$ by permutation of interior marked points. Next we choose an obstruction bundle $\mathbb{E}$ over $\mathcal{U}$ as follows (see [15, Definitions 17.7, 17.15]). We take a Γ-equivariant smooth family of finite dimensional subspaces $\mathbb{E}_{\mathfrak{a}}$

$$\mathbb{E}_{\mathfrak{a}}\subset C^\infty_c(\mathrm{Int}(K),u_0^*TM\otimes\Lambda^{0,1})$$

parametrized by $\mathfrak{a}=(\Sigma,x,\{w_1,\dots,w_l\})\in N$, where $K=\Sigma\cap\mathcal{K}$ is the core of Σ and $\Lambda^{0,1}$ is the bundle of $(0,1)$-forms on Σ. Then we extend this family to the whole $\mathcal{U}$ via parallel transport, i.e. for each point $\mathfrak{x}=(u,\Sigma,x,\{w_1,\dots,w_l\})\in\mathcal{U}$ over $\mathfrak{a}=(\Sigma,x,\{w_1,\dots,w_l\})\in N$, we define

$$\mathbb{E}_{\mathfrak{x}}\subset C^\infty_c(\mathrm{Int}(K),u^*TM\otimes\Lambda^{0,1})$$

as the parallel transport of $\mathbb{E}_{\mathfrak{a}}$ along geodesics joining $u(y)$ and $u_0(y)$. Here we use a connection on TM such that TL is preserved by parallel translation [15, §11]. By construction, the bundle $\mathbb{E}\to\mathcal{U}$ is Γ-equivariant. The Kuranishi neighbourhood $V\subset\mathcal{U}$ is now cut out by the equations:

$$\begin{aligned}&\overline{\partial}u\equiv 0\mod\mathbb{E}_{\mathfrak{x}}\\&u(w_i)\in Q_i\quad i=1,\dots,l\end{aligned}\tag{20}$$

for $\mathfrak{x}=(u,\Sigma,x,\{w_1,\dots,w_l\})\in\mathcal{U}$. We need to choose $\mathbb{E}$ so that the equations (20) are transversal (see below). The Γ-action on $\mathcal{U}$ preserves V and the obstruction bundle restricts to a Γ-equivariant vector bundle $E=\mathbb{E}|_V$ over V. The Cauchy-Riemann operator $\overline{\partial}$ induces a section s of $E\to V$ and $s^{-1}(0)/\Gamma$ gives a neighbourhood of $\mathfrak{x}_0\in\mathcal{M}_1(\alpha)$.

The required transversality for (20) is stated as follows (see [15, Lemmata 18.16, 20.7]). For a smooth map $u\colon(\Sigma,\partial\Sigma)\to(M,L)$, let $L^2_{m,\delta}(\Sigma,\partial\Sigma;u^*TM,u^*TL)$ denote a certain weighted Sobolev space consisting of $L^2_{m,\mathrm{loc}}$-sections of u^*TM which take values in u^*TL along the boundary $\partial\Sigma$, see [15, Definitions 10.1, 19.8] (m is sufficiently large and $\delta>0$ is a parameter relevant to the weighted Sobolev norm). Let $L^2_{m,\delta}(\Sigma,u^*TM\otimes\Lambda^{0,1})$ denote a similar weighted Sobolev space of sections of $u^*TM\otimes\Lambda^{0,1}$ (see [15, Definition 19.9]). Let $D_{\mathfrak{x}}\overline{\partial}$ denote the linearized operator of $\overline{\partial}$ at $\mathfrak{x}=(u,\Sigma,x,\{w_1,\dots,w_l\})\in\mathcal{U}$:

$$D_{\mathfrak{x}}\overline{\partial}\colon L^2_{m+1,\delta}(\Sigma,\partial\Sigma;u^*TM,u^*TL)\to L^2_{m,\delta}(\Sigma,u^*TM\otimes\Lambda^{0,1})$$

where the connection on TM is used to define the derivative $D_{\mathfrak{x}}\overline{\partial}$ (see [15, Remark 12.5]). We require that $\mathrm{Im}(D_{\mathfrak{x}}\overline{\partial})$ and $\mathbb{E}_{\mathfrak{x}}$ span $L^2_{m,\delta}(\Sigma,u^*TM\otimes\Lambda^{0,1})$ for each $\mathfrak{x}\in\mathcal{U}$. (This is called "Fredholm regularity".) Let $\mathcal{M}\subset\mathcal{U}$ denote the subspace cut out only by the first equation of (20). Let $\mathrm{ev}_{\mathrm{ad}}\colon\mathcal{M}\to M^l$ be the evaluation map at the l additional markings. We also require that $\mathrm{ev}_{\mathrm{ad}}$ is transversal to $\prod_{i=1}^l Q_i\subset M^l$. Then we have $V=\mathrm{ev}_{\mathrm{ad}}^{-1}(\prod_{i=1}^l Q_i)$.

Now we construct a Kuranishi neighbourhood of $\mathfrak{f}^{-1}(\mathfrak{x}_0\times S^1)\subset\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ from the Kuranishi neighbourhood (V,E,Γ,ψ,s) of $\mathfrak{x}_0\in\mathcal{M}_1(\alpha)$ above. Recall that $\mathfrak{f}^{-1}(\mathfrak{x}_0\times S^1)\cong\widehat{\Sigma}_0\times S^1$

by Lemma 3.12, where $\widehat{\Sigma}_0$ is the oriented real blow-up of Σ_0 at x_0. Here we perform oriented real blow-ups in families. The family $\mathcal{R} \to N$ is equipped with a section $x\colon N \to \mathcal{R}$ corresponding to the boundary marked point. Let $\widehat{\mathcal{R}}$ denote the oriented real-blow up along the section x. The proof of Lemma 3.12 shows that a point $p \in \widehat{\mathcal{R}}$ parametrizes a marked stable bordered Riemann surface[4] $(\widetilde{\Sigma}, x, p, \{w_1, \dots, w_l\})$ with a new interior marking p (see [12, Lemma 7.1.45]). More precisely, letting $p \in \widehat{\mathcal{R}}$ be on the blow-up of a fibre $\Sigma \subset \mathcal{R}$: if p is neither a node nor a boundary point, $\widetilde{\Sigma} = \Sigma$; if p is an interior node, $\widetilde{\Sigma}$ is obtained from Σ by adding a sphere bubble at the node; if p is a boundary point, $\widetilde{\Sigma}$ is obtained from Σ by adding at most two disc bubbles (see Figure 1). We allow p to coincide with one of w_i's. Let $\mathfrak{R} \to \widehat{\mathcal{R}}$ denote the corresponding family of marked stable bordered Riemann surfaces. The group Γ acts on $\mathfrak{R}$ by permutation of the markings $w_1, \dots, w_l$. The core $\mathcal{K} \subset \mathcal{R}$ canonically induces a Γ-invariant core $\mathfrak{K} \subset \mathfrak{R}$ equipped with a Γ-equivariant C^∞-trivialization $\mathfrak{K} \cong K_0 \times \widehat{\mathcal{R}}$. Here $\mathrm{Int}(\mathfrak{K})$ is disjoint from the components contracted under $\widetilde{\Sigma} \to \Sigma$. We consider the space $\mathfrak{U}$ consisting of tuples $(\hat{u}, \widetilde{\Sigma}, x, p, \{w_1, \dots, w_l\})$ where $(\widetilde{\Sigma}, x, p, \{w_1, \dots, w_l\})$ is a marked bordered Riemann surface corresponding to a point of $\widehat{\mathcal{R}}$ (i.e. arises as a fibre of $\mathfrak{R} \to \widehat{\mathcal{R}}$) and $\hat{u}\colon (\widetilde{\Sigma}, \partial\widetilde{\Sigma}) \to (M \times \mathbb{P}^1, L \times S^1)$ is a smooth map of class $\hat{\alpha}$ which satisfies the following conditions:

- $\pi_M \circ \hat{u}$ is C^{10}-close to u_0 on the core $\widetilde{K} = \widetilde{\Sigma} \cap \mathfrak{K}$, where $\pi_M\colon M \times \mathbb{P}^1 \to M$ is the projection (since $\widetilde{K}$ is identified with K_0 via the given trivialization $\mathfrak{K} \cong K_0 \times \widehat{\mathcal{R}}$, u_0 defines a map $u_0\colon \widetilde{K} \to M$);
- $\hat{u}$ is holomorphic on the neck region $\widetilde{\Sigma} \setminus \widetilde{K}$;
- the diameter of the image of each connected component of the neck region under $\pi_M \circ \hat{u}$ is small.

We use the obstruction bundle $\mathbf{E} \to \mathfrak{U}$ induced from $\mathbb{E} \to \mathcal{U}$ as follows. Take an element $\mathfrak{s} = (\hat{u}, \widetilde{\Sigma}, x, p, \{w_1, \dots, w_l\}) \in \mathfrak{U}$ and let $\mathfrak{a} = (\Sigma, x, \{w_1, \dots, w_l\}) \in N$ denote the marked Riemann surface given by forgetting p and collapsing unstable components of the source. Define the obstruction space at $\mathfrak{s} \in \mathfrak{U}$

$$\mathbf{E}_{\mathfrak{s}} \subset C_c^\infty(\mathrm{Int}(\widetilde{K}), (\pi_M \circ \hat{u})^* TM \otimes \Lambda^{0,1}) \subset C_c^\infty(\mathrm{Int}(\widetilde{K}), \hat{u}^* T(M \times \mathbb{P}^1) \otimes \Lambda^{0,1})$$

(with $\widetilde{K} = \widetilde{\Sigma} \cap \mathfrak{K}$) to be the parallel transport of $\mathbb{E}_{\mathfrak{a}} \subset C_c^\infty(\mathrm{Int}(\widetilde{K}), u_0^* TM \otimes \Lambda^{0,1})$ along geodesics joining $u_0(y)$ and $(\pi_M \circ \hat{u})(y)$. Let $C \subset \widetilde{\Sigma}$ be the contracted components of $\widetilde{\Sigma} \to \Sigma$. Because $\pi_M \circ \hat{u}|_{\widetilde{K}}$ is sufficiently close to u_0 and is holomorphic on C, by choosing a smaller neck region from the beginning if necessary (see "extending the core" [15, Definition 17.21]), we may assume that $\pi_M \circ \hat{u}$ is constant on C (since the symplectic area of the neck region has to be small). Hence $\pi_M \circ \hat{u}$ induces a map $u\colon (\Sigma, \partial\Sigma) \to (M, L)$ belonging to $\mathcal{U}$. Therefore we have a projection $\mathfrak{U} \to \mathcal{U}$ and $\mathbf{E}$ is identified with the pull-back of $\mathbb{E}$. The group Γ acts on $\mathfrak{U}$ and $\mathbf{E}$ and $\mathbf{E} \to \mathfrak{U}$ is Γ-equivariant. The Kuranishi neighbourhood $\widehat{V}$ for

[4]By abuse of notation, we denote by p a point of $\widehat{\mathcal{R}}$ and at the same time a new interior marking on $\widetilde{\Sigma}$.

$\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$ is cut out from $\mathfrak{U}$ by the equations:

$$\begin{aligned} &\overline{\partial}\hat{u} \equiv 0 \mod \mathbf{E}_{\mathfrak{s}} \\ &\hat{u}(p) \in M \times \{0\} \\ &\hat{u}(w_i) \in Q_i \times \mathbb{P}^1 \quad i = 1, \dots, l \end{aligned} \tag{21}$$

for $\mathfrak{s} = (\hat{u}, \widetilde{\Sigma}, x, p, \{w_1, \dots, w_l\}) \in \mathfrak{U}$. The second equation of (21) corresponds to the fibre product presentation (10) of $\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$. Let $\widehat{\mathcal{M}} \subset \mathfrak{U}$ denote the subspace cut out by the first and the second equations of (21). Consider the map $\mathfrak{U} \to \mathcal{U} \times S^1$, where the first factor is the projection we discussed and the second factor is the evaluation map at the boundary marking x followed by the projection $L \times S^1 \to S^1$. We claim that $\widehat{\mathcal{M}}$ is a tautological family of (blown-up) Riemann surfaces over $\mathcal{M} \times S^1$ under the map $\widehat{\mathcal{M}} \subset \mathfrak{U} \to \mathcal{U} \times S^1$. (Recall that $\mathcal{M} \subset \mathcal{U}$ is cut out by the first equation of (20).) More precisely, it is identified with the restriction to $\mathcal{M} \times S^1$ of the family $\mathrm{pr}^* \widehat{\mathcal{R}} \to \mathcal{U} \times S^1$ where $\mathrm{pr} \colon \mathcal{U} \times S^1 \to N \times S^1$ is the natural projection. By the choice of $\mathbf{E}$, each element $(\hat{u}, \widetilde{\Sigma}, x, p, \{w_1, \dots, w_l\})$ of $\widehat{\mathcal{M}}$ is holomorphic in the $\mathbb{P}^1$-factor and its image $(u, \Sigma, x, \{w_1, \dots, w_l\})$ in $\mathcal{U}$ belongs to $\mathcal{M}$. By the same argument as in the proof of Lemma 3.12, it follows that $\hat{u}$ is uniquely reconstructed from $u \colon \Sigma \to M$, $p \in \widehat{\Sigma}$ and $(\pi_{\mathbb{P}^1} \circ \hat{u})(x) \in S^1$. This proves the claim. Cutting down the moduli space $\widehat{\mathcal{M}}$ by the third equation of (21), we obtain $\widehat{V}$ as a tautological family of (blown-up) Riemann surfaces over $V \times S^1$, with V the Kuranishi neighbourhood of $\mathfrak{r}_0 \in \mathcal{M}_1(\alpha)$. The obstruction bundle $\widehat{E} = \mathbf{E}|_{\widehat{V}}$ and its section $\hat{s} := \overline{\partial}$ are the pull-backs of $E \to V$ and $s = \overline{\partial}$ respectively. These data $(\widehat{V}, \widehat{E}, \hat{s})$ are Γ-equivariant and give a Kuranishi neighbourhood $(\widehat{V}, \widehat{E}, \Gamma, \hat{\psi}, \hat{s})$ of $\mathfrak{f}^{-1}(\mathfrak{r}_0 \times S^1) \subset \mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})$.

We need to check the transversality of (21). To see the transversality of the $\overline{\partial}$-equation, it suffices to show that, for a map $\hat{u}$ satisfying the first equation of (21), the degree-one holomorphic map $v := \pi_{\mathbb{P}^1} \circ \hat{u} \colon (\widetilde{\Sigma}, \partial\widetilde{\Sigma}) \to (\mathbb{P}^1, S^1)$ is Fredholm regular (since $\pi_M \circ \hat{u}$ is already known to be Fredholm regular with respect to $\mathbb{E}$). The Fredholm regularity here can be rephrased as the vanishing of sheaf cohomology (see [21, §3.4], [9, §6]):

$$H^1(\widetilde{\Sigma}, (v^*T\mathbb{P}^1, v^*TS^1)) = 0$$

where $(v^*T\mathbb{P}^1, v^*TS^1)$ denotes the sheaf of holomorphic sections of $v^*T\mathbb{P}^1$ which take values in v^*TS^1 on $\partial\widetilde{\Sigma}$. We can prove this by using the standard normalization sequence and [9, Lemma 6.4]. Let $\mathcal{N} \subset \mathfrak{U}$ denote the moduli space cut out only by the first equation of (21). The holomorphic automorphism group $\mathrm{Aut}(\mathbb{D})$ acts on the target $(M \times \mathbb{P}^1, L \times S^1)$ and also on the moduli space $\mathcal{N}$. The transversality for the second equation of (21) follows from the fact that the $\mathrm{Aut}(\mathbb{D})$-action on $\mathrm{Int}(\mathbb{D})$ is transitive. The first and the second equations of (21) define the modui space $\widehat{\mathcal{M}}$. The evaluation map $\mathrm{ev}_{\mathrm{ad}} \colon \widehat{\mathcal{M}} \to (M \times \mathbb{P}^1)^l$ at the markings $w_1, \dots, w_l$ is transversal to $\prod_{i=1}^l (Q_i \times \mathbb{P}^1)$ by the transversality assumption for the second equation of (20). The transversality for (21) follows.

Finally we compare virtual cycles. A virtual cycle is defined by multi-valued perturbations (multisections) of s on Kuranishi neighbourhoods which are compatible under co-ordinate changes (see [12, §A1.1], [15, Part 2]). By the above construction of Kuranishi neighbourhoods, we can define a virtual cycle $[\mathcal{M}^{\mathrm{rel}}_{1,1}(\hat{\alpha})]^{\mathrm{vir}}$ by pulling back multisections used to define

a virtual cycle $[\mathcal{M}_1(\alpha)]^{\mathrm{vir}}$. Then $[\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})]^{\mathrm{vir}}$ becomes a fibre bundle over $[\mathcal{M}_1(\alpha)]^{\mathrm{vir}} \times S^1$ with fibre the corresponding stable bordered Riemann surfaces. Each fibre is of class α under the interior evaluation map. The lemma follows. $\square$

Summarizing the discussion, we obtain (see (14), (18), (19)):

Lemma 3.16. *The virtual cycle $\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})]^{\mathrm{vir}}$ is well-defined if one of the following holds:*

(a) $\mathcal{M}_{\mathrm{S}}(\sigma) = \emptyset$ *or;*
(b) $\mathcal{M}_1(\alpha) = \emptyset$ *or;*
(c) $\partial\mathcal{M}_1(\alpha) = \emptyset$ *and* $\mathrm{ev}(\mathcal{M}_{\mathrm{S}}(\sigma)) \cap L = \emptyset$.

When one of the above conditions holds, we have:

$$\mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma)\times_M\mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})]^{\mathrm{vir}} = \begin{cases} (\mathcal{S}_\sigma \cap [L]) \times [S^1] & \text{if } \alpha = 0 \text{ (then (b) holds);} \\ \langle \widehat{\mathcal{S}}_\sigma, \alpha\rangle \, \mathrm{ev}_*[\mathcal{M}_1(\alpha)]^{\mathrm{vir}} \times [S^1] & \text{if (c) holds;} \\ 0 & \text{if } \alpha \neq 0 \text{ and (a) or (b) holds.} \end{cases}$$

3.3.3. *Conjecture and expected results.* We now state our conjecture:

Conjecture 3.17 (Degeneration Formula)**.** *Let $\beta \in H_2(M, L)$ be such that $\mathcal{M}_1(\beta) = \emptyset$. Assume that every pair $(\sigma, \alpha) \in H_2^{\mathrm{sec}}(E) \times H_2(M, L)$ with $r(\hat{\beta}) = \sigma + \hat{\alpha}$ satisfies one of the three conditions (a), (b), (c) in Lemma 3.16. Then the degeneration formula* (11)

$$\varphi_* \, \mathrm{ev}_*[\mathcal{M}_1(\hat{\beta})]^{\mathrm{vir}} = \sum_{r(\hat{\beta}) = \sigma + \hat{\alpha}} \mathrm{ev}_*[\mathcal{M}_{\mathrm{S}}(\sigma) \times_M \mathcal{M}_{1,1}^{\mathrm{rel}}(\hat{\alpha})]^{\mathrm{vir}}$$

holds. This implies, by Lemmata 3.11 and 3.16, that

$$\delta_{\beta,0}[L] = \sum_{\substack{(\sigma,\alpha):\, r(\hat{\beta}) = \sigma + \hat{\alpha},\ \alpha \neq 0 \\ \text{satisfying (c) of Lemma 3.16}}} \langle \widehat{\mathcal{S}}_\sigma, \alpha \rangle \, \mathrm{ev}_*[\mathcal{M}_1(\alpha)]^{\mathrm{vir}} + \delta_{\partial\beta,\lambda} \sum_{\sigma: r(\hat{\beta}) = \sigma + \hat{0}} \mathcal{S}_\sigma \cap [L] \tag{22}$$

holds in $H_{n+\mu(\beta)}(L; \mathbb{Q})$. Here $\mathcal{S}_\sigma$, $\widehat{\mathcal{S}}_\sigma$ are defined in (15), (17) *and $\lambda \in H_1(L)$ is the class of an S^1-orbit.*

Note that the second term in the right-hand side of (22) arises from the case $\alpha = 0$ (recall the discussion around (14)).

In practice it is not easy to make all the assumptions here to be satisfied and to obtain a non-trivial result from (22). Notice that the both-hand sides of (22) are zero unless $\mu(\beta) \leq 0$ for dimensional reason. Also the term $\langle \widehat{\mathcal{S}}_\sigma, \alpha \rangle$ is zero unless $\widehat{\mathcal{S}}_\sigma \in H^2(M, L)$, i.e. $\langle c_1^{\mathrm{vert}}(E), \sigma \rangle = -1$. Hence by (12), the first term of the right-hand side is the sum over classes α satisfying $\mu(\alpha) = \mu(\beta) + 2$. This motivates the following (rather restrictive) assumption:

Assumption 3.18. *(i) $\mathcal{M}_1(\beta)$ is empty for all $\beta \in H_2(M, L)$ with $\mu(\beta) \leq 0$.*

(ii) The maximal fixed component $F_{\max} \subset M$ of the $\mathbb{C}^\times$-action (see §2.2) is of complex codimension one and the $\mathbb{C}^\times$-weight on the normal bundle is -1.

(iii) $c_1(M)$ is semi-positive.

(iv) $\mathrm{ev}(\mathcal{M}_{\mathrm{S}}(\sigma))$ is disjoint from L for all $\sigma \in H_2^{\mathrm{sec}}(E)$ such that $\langle c_1^{\mathrm{vert}}(E), \sigma \rangle = -1$.

We assume Assumption 3.18 in the rest of this section. Recall from Definition 2.1 that open Gromov-Witten invariants n_α are defined when $\mu(\alpha) = 2$ by the assumption (i) and so the potential function W of L is also defined. The role of the assumptions (ii) and (iii) is as follows. The assumption (ii) implies that $\langle c_1^{\mathrm{vert}}(E), \sigma_0\rangle = -1$ for a maximal section σ_0. Note that by (3) $\mathcal{M}_{\mathrm{S}}(\sigma)$ is empty unless $\sigma = \sigma_0 + d$ for some $d \in \mathrm{NE}(M)_{\mathbb{Z}}$. Therefore by (iii), $\mathcal{M}_{\mathrm{S}}(\sigma)$ is empty unless $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle \ge -1$. This implies that the Seidel element S in Definition 2.7 is in $H^{\le 2}(M;\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{NE}(E)_{\mathbb{Z}}]\!]$.

Definition 3.19. Under Assumption 3.18, we can decompose the Seidel element as

$$S = q_0 \widetilde{S} = q_0(\widetilde{S}^{(0)} + \widetilde{S}^{(2)})$$

with $\widetilde{S}^{(i)} \in H^i(M;\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{NE}(M)_{\mathbb{Z}}]\!]$ and $q_0 = q^{\sigma_0}$. Furthermore, we can define a lift $\widehat{S}^{(2)}$ of $\widetilde{S}^{(2)}$ as follows:

$$\widehat{S}^{(2)} := \sum_{\sigma\colon \langle c_1^{\mathrm{vert}}(E),\sigma\rangle = -1} \widehat{\mathcal{S}}_\sigma q^{\sigma-\sigma_0}$$

where $\widehat{\mathcal{S}}_\sigma \in H^2(M,L;\mathbb{Q})$ (see (17)) is well-defined by Assumption 3.18 (iv). The lift $\widehat{S}^{(2)}$ is an element of $H^2(M,L;\mathbb{Q}) \otimes \mathbb{Q}[\![\mathrm{NE}(M)_{\mathbb{Z}}]\!]$ which maps to $\widetilde{S}^{(2)}$ under the natural map $H^2(M,L) \to H^2(M)$.

Under Assumption 3.18, the conditions in Conjecture 3.17 are satisfied for all β with $\mu(\beta) = 0$. In fact, if $r(\hat{\beta}) = \sigma + \hat{\alpha}$, then $\mu(\alpha) + 2\langle c_1^{\mathrm{vert}}(E), \sigma\rangle = 0$ by (12), and thus

- if $\mu(\alpha) \le 0$ and $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle \ge 0$, then $\mathcal{M}_1(\alpha) = \emptyset$ by the assumption (i);
- if $\mu(\alpha) \ge 4$ and $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle \le -2$, then $\mathcal{M}_{\mathrm{S}}(\sigma) = \emptyset$ by the assumptions (ii), (iii);
- if $\mu(\alpha) = 2$ and $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle = -1$, then $\mathcal{M}_1(\alpha)$ has no boundary by the assumption (i) and $\mathrm{ev}(\mathcal{M}_{\mathrm{S}}(\sigma)) \cap L = \emptyset$ by the assumption (iv).

Fix a class $\gamma \in H_1(L)$. We now apply the formula (22) for β with $\mu(\beta) = 0$ and $\partial\beta = \gamma + \lambda$. In this case, (22) yields the following equality in $H_n(L;\mathbb{Q}) \cong \mathbb{Q}$:

$$\delta_{\beta,0} = \sum_{\substack{(\sigma,\alpha)\colon r(\hat{\beta}) = \sigma+\hat{\alpha} \\ \mu(\alpha)=2,\ \langle c_1^{\mathrm{vert}}(E),\sigma\rangle = -1}} \langle \widehat{\mathcal{S}}_\sigma, \alpha\rangle n_\alpha + \delta_{\partial\beta,\lambda} \sum_{\sigma\colon r(\hat{\beta}) = \sigma + \hat{0}} \mathcal{S}_\sigma \tag{23}$$

where n_α is the open Gromov-Witten invariant defined in Definition 2.1. Note that $\mathcal{S}_\sigma$ in the second term of the right-hand side lies in $H^0(L;\mathbb{Q}) \cong \mathbb{Q}$. We consider a generating function in the "open" Novikov ring Λ^{op} which was introduced before Definition 2.1. We have a (not necessarily injective) homomorphism from the "closed" Novikov ring Λ (see Remark 2.9) to the "open" Novikov ring Λ^{op}

$$\Lambda \to \Lambda^{\mathrm{op}}, \quad q^d \mapsto z^d.$$

Thus Λ^{op} is a Λ-algebra. Note that $r(\hat{\beta}) = \sigma + \hat{\alpha}$ means

$$z^{\alpha_0+\beta} = q^{\sigma-\sigma_0} z^\alpha \quad \text{in } \Lambda^{\mathrm{op}}$$

by Proposition 3.6 where σ_0, α_0 are maximal section/disc classes. We multiply the both-hand sides of (23) by $z^{\alpha_0+\beta} = q^{\sigma-\sigma_0} z^\alpha$ and sum over all β with $\mu(\beta) = 2$ and $\partial\beta = \gamma + \lambda$. About the first term of the right-hand side, this summation boils down to the sum over all (σ,α) with $\langle c_1^{\mathrm{vert}}(E), \sigma\rangle = -1$, $\mu(\alpha) = 2$, $\partial\alpha = \gamma$ (see (13)); about the second term of the right-hand

side (which occurs when and only when $\gamma = 0$), this boils down to the sum over all σ with $\langle c_1^{\mathrm{vert}}(E), \sigma \rangle = 0$. Therefore we have:

Theorem 3.20. *Assume that the degeneration formula (Conjecture 3.17) and Assumption 3.18 hold for (M, L). For any $\gamma \in H_1(L)$, we have*

$$\delta_{\gamma+\lambda,0} z^{\alpha_0} = \langle \widehat{S}^{(2)}, dW_\gamma \rangle + \delta_{\gamma,0} \widetilde{S}^{(0)} \tag{24}$$

in Λ^{op}, where $\widetilde{S}^{(0)}$ and $\widehat{S}^{(2)}$ are in Definition 3.19, W_γ is in Definition 2.2 and dW_γ is its logarithmic derivative:

$$dW_\gamma := \sum_{\alpha \in H_2(M,L) \colon \mu(\alpha)=2,\ \partial\alpha=\gamma} \alpha \otimes n_\alpha z^\alpha \in H_2(M, L) \otimes \Lambda^{\mathrm{op}}.$$

Recall that α_0 is the maximal disc class introduced before Proposition 3.6 and $\lambda \in H_1(L)$ is the class of an S^1-orbit on L.

Summing over all $\gamma \in H_1(L)$ in (24), we obtain:

Corollary 3.21. *Assume that the degeneration formula (Conjecture 3.17) and Assumption 3.18 hold for (M, L). Then we have*

$$z^{\alpha_0} = \langle \widehat{S}^{(2)}, dW \rangle + \widetilde{S}^{(0)} \quad \textit{in } \Lambda^{\mathrm{op}}. \tag{25}$$

Via the natural map $H^1(L) \to H^2(M, L)$, an element of $H^1(L)$ can be regarded as a vector field tangent to the fibre of the map $\operatorname{Spec} \Lambda^{\mathrm{op}} \to \operatorname{Spec} \Lambda$. We define the (relative) *Jacobi algebra* of the potential W as

$$\mathrm{Jac}(W) := \Lambda^{\mathrm{op}} / \Lambda^{\mathrm{op}} \langle H^1(L), dW \rangle$$

where $\Lambda^{\mathrm{op}} \langle H^1(L), dW \rangle$ denotes the ideal of Λ^{op} generated by $\langle \varphi, dW \rangle$, $\varphi \in \mathrm{Im}(H^1(L) \to H^2(M, L))$. As a class in the Jacobi algebra, the right-hand side of (25) depends only on the Seidel element $\widetilde{S}$ itself, not on the lift $\widehat{S}^{(2)}$. We can interpret it as the derivative of the bulk-deformed potential $W + t^0$ with respect to $\widetilde{S}$, where t^0 is a co-ordinate on $H^0(M)$. The derivative of $W + t^0$ defines the so-called *Kodaira-Spencer mapping*:

$$\mathrm{KS} \colon H^{\leq 2}(M) \otimes \Lambda \to \mathrm{Jac}(W).$$

Then the equation (25) implies

$$\mathrm{KS}(\widetilde{S}) = [z^{\alpha_0}] \qquad \text{in} \quad \mathrm{Jac}(W).$$

Remark 3.22. Assumption 3.18 (i)–(iii) ensures that the conditions in Conjecture 3.17 hold for all β with $\mu(\beta) \leq -2$. Using the formula (22) for β with $\mu(\beta) \leq -2$ and $\partial\beta = \lambda$, we find:

$$\sum_{d \in i_* H_2(L)} \mathcal{S}_{\sigma+d} \cap [L] = 0 \quad \text{if} \quad \left\langle c_1^{\mathrm{vert}}(E), \sigma \right\rangle \leq -1.$$

This supports the validity of Assumption 3.18 (iv).

Remark 3.23. A more intuitive explanation for the formula (25) is as follows. One can think of the moduli space $\mathcal{M}_{1,1}(\beta)$ of stable holomorphic discs with boundaries in L and with one interior and one boundary marked points as giving a correspondence between M and the

free loop space $\mathcal{L}L = \mathrm{Map}(S^1, L)$ of L. This correspondence should give rise to a map (bulk-boundary map)

$$C_*(M) \to C_*(\mathcal{L}L)$$

of chain complexes. One can view this as an analogue of the Kodaira-Spencer map. One can speculate that this map is an intertwiner between the Seidel homomorphism $S \colon C_*(M) \to C_*(M)$ and the map $\mathcal{L}L \to \mathcal{L}L$ induced by the S^1-action.

4. Potential function of a semi-positive toric manifold

Using the degeneration formula (Conjecture 3.17), we compute the potential function of a Lagrangian torus fibre of a semi-positive toric manifold X. This confirms a conjecture (now a theorem [8]) of Chan-Lau-Leung-Tseng [7].

4.1. **Toric manifolds.** We fix notation on toric geometry. For more details we refer the reader to [1, 10, 11]. For this paper a *toric manifold* X is a smooth projective toric variety, as constructed from the following data.

(a) An integral lattice $N \cong \mathbb{Z}^n$ and its dual $M = \mathrm{Hom}(N, \mathbb{Z})$. We denote by $\langle \cdot, \cdot \rangle$ the natural pairing between N and M.
(b) A fan Σ in $N_\mathbb{R} := N \otimes \mathbb{R}$ consisting of a collection of strongly convex rational polyhedral cones $\sigma \subset N_\mathbb{R}$, which is closed under intersections and taking faces.

In order for X to be smooth and projective, we need to assume that Σ is complete, regular and admits a strongly convex piecewise-linear function. Let $\Sigma(1)$ denote the set of 1-cones (rays) in Σ, and we let $b_1, \dots, b_m$ denote integral primitive generators of the 1-cones. The *fan sequence* of X is the exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^m \longrightarrow N \longrightarrow 0, \tag{26}$$

where the third arrow takes the canonical basis to the primitive generators $b_1, \dots, b_m \in N$ and $\mathbb{L}$ is defined to be the kernel of the third arrow. The dual of the sequence (26) is the *divisor sequence*

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^m \xrightarrow{\ \kappa\ } \mathbb{L}^\vee \longrightarrow 0. \tag{27}$$

The second arrow takes $v \in M$ into the tuple $(\langle b_i, v \rangle)_{i=1}^m$. The third arrow is denoted by $\kappa \colon \mathbb{Z}^m \to \mathbb{L}^\vee$.

The fan sequence tensored with $\mathbb{C}^\times$ gives the exact sequence of tori:

$$1 \longrightarrow \mathbb{G} \longrightarrow (\mathbb{C}^\times)^m \longrightarrow \mathbb{T} \longrightarrow 1$$

with $\mathbb{G} := \mathbb{L} \otimes \mathbb{C}^\times$ and $\mathbb{T} := N \otimes \mathbb{C}^\times$. Let the torus $\mathbb{G}$ act on $\mathbb{C}^m$ by the second arrow $\mathbb{G} \to (\mathbb{C}^\times)^m$. The combinatorics of the fan defines a stability condition of this action as follows. Let $Z(\Sigma)$ denote the union

$$Z(\Sigma) := \bigcup_{I \in \mathcal{A}} \mathbb{C}^I, \quad \mathbb{C}^I = \{(x_1, \dots, x_m) : x_i = 0 \text{ for } i \notin I\}, \tag{28}$$

where $\mathcal{A}$ is the collection of anti-cones, that is the subsets of indices that do not yield a cone in the fan

$$\mathcal{A} := \left\{ I \ : \ \sum_{i \in I} \mathbb{R}_{\geq 0} b_i \notin \Sigma \right\}.$$

The toric variety X is defined as the quotient

$$X := \mathcal{U}_\Sigma/\mathbb{G}; \quad \mathcal{U}_\Sigma := \mathbb{C}^m \setminus Z(\Sigma).$$

The torus $\mathbb{T} = (\mathbb{C}^\times)^m/\mathbb{G}$ acts naturally on X. The toric manifold X contains $\mathbb{T}$ as an open free orbit; X is a compactification of $\mathbb{T}$ along the rays in $\Sigma(1)$.

Each character $\xi : \mathbb{G} \to \mathbb{C}^\times$ defines a line bundle

$$L_\xi := \mathbb{C} \times_{\xi,\mathbb{G}} \mathcal{U}_\Sigma \to X.$$

The correspondence $\xi \mapsto L_\xi$ yields an identification of the Picard group with the character group of $\mathbb{G}$. Thus, we have

$$\mathbb{L}^\vee = \mathrm{Hom}(\mathbb{G}, \mathbb{C}^\times) \cong \mathrm{Pic}(X) \overset{c_1}{\cong} H^2(X;\mathbb{Z}).$$

The ith toric divisor is given by

$$D_i := \{[x_1, \dots, x_m] \,:\, x_i = 0\} \subset X$$

The Poincaré dual of D_i is the image $\kappa(e_i) \in \mathbb{L}^\vee \cong H^2(X;\mathbb{Z})$ of the standard basis $e_i \in \mathbb{Z}^m$ under the map κ in (27). By abuse of notation, D_i sometimes also denotes the corresponding cohomology class $\kappa(e_i)$ in $H^2(X;\mathbb{Z})$. We note that $\mathbb{L} = H_2(X;\mathbb{Z})$. The first Chern class $c_1(X)$ of X is given by $D_1 + \cdots + D_m$.

The *Kähler cone* C_X of X, the cone consisting of Kähler classes, is given by

$$C_X := \bigcap_{I \in \mathcal{A}} \sum_{i \in I} \mathbb{R}_{>0}\kappa(e_i) \subset \mathbb{L}^\vee \otimes \mathbb{R} = H^2(X;\mathbb{R}).$$

The cone C_X is nonempty if and only if X is projective. Set $r := m - n$. We choose a nef integral basis $p_1, \dots, p_r$ of $H^2(X;\mathbb{Z})$, that is an integral basis such that $p_a \in \overline{C}_X$ for all $a = 1, \dots, r$. Then we write the toric divisor classes as

$$D_j = \kappa(e_j) = \sum_{a=1}^{r} m_{aj} p_a, \tag{29}$$

for some integer matrix (m_{aj}). The *Mori cone* $\mathrm{NE}(X) \subset H_2(X,\mathbb{R})$ is the dual of the cone $\overline{C}_X$. We set $\mathrm{NE}(X)_\mathbb{Z} := \mathrm{NE}(X) \cap H_2(X;\mathbb{Z})$.

The toric manifold X can be alternatively defined as a symplectic quotient. Let $\mathbb{G}_\mathbb{R} \cong (S^1)^r$ be the maximal compact subgroup in $\mathbb{G}$. The $\mathbb{G}_\mathbb{R}$-action on $\mathbb{C}^m$ is generated by the moment map

$$\phi\colon \mathbb{C}^m \to \mathfrak{g}_\mathbb{R}^\vee, \quad \phi(x_1, \dots, x_m) = \kappa(|x_1|^2, \dots, |x_m|^2)$$

where $\kappa\colon \mathbb{R}^m \to \mathbb{L}^\vee \otimes \mathbb{R}$ is the map in the divisor sequence (27) tensored with $\mathbb{R}$. For any Kähler class $\omega \in C_X$, we have a diffeomorphism ([1, 20])

$$\phi^{-1}(\omega)/\mathbb{G}_\mathbb{R} \cong X.$$

The left-hand side is a symplectic quotient and is equipped with a reduced symplectic form. The cohomology *class* of the reduced symplectic form coincides with ω; by abuse of notation we let ω also denote the reduced symplectic *form*.

Let $\mathbb{T}_{\mathbb{R}} \cong (S^1)^n$ be the maximal compact subgroup of $\mathbb{T}$. The $\mathbb{T}_{\mathbb{R}}$-action on the symplectic toric manifold (X, ω) admits a moment map:

$$\Phi_\omega \colon X \longrightarrow \kappa^{-1}(\omega),$$
$$\Phi_\omega([x_1, \dots, x_m]) = (|x_1|^2, \dots, |x_m|^2) \quad \text{with} \quad (x_1, \dots, x_m) \in \phi^{-1}(\omega).$$

where the affine subspace $\kappa^{-1}(\omega) \subset \mathbb{R}^m$ can be identified with $M_{\mathbb{R}} = M \otimes \mathbb{R} \cong \mathfrak{t}_{\mathbb{R}}^\vee$ up to translation. The image of the moment map Φ_ω is the convex polytope:

$$\begin{aligned} P(\omega) &= \{(t_1, \dots, t_m) \in \mathbb{R}^m \ : \ t_i \geq 0, \ \kappa(t_1, \dots, t_m) = \omega\} \\ &\cong \{v \in M_{\mathbb{R}} \ : \ \langle b_i, v\rangle \leq -c_i, \ i = 1, \dots, m\}. \end{aligned}$$

In the second line, we took a lift $(c_1, \dots, c_m)$ of ω (such that $\omega = \kappa(c_1, \dots, c_m)$) to identify $\kappa^{-1}(\omega)$ with $M_{\mathbb{R}}$. This is called *momentum polytope*. The facet $F_i \subset P$ of $P(\omega)$ normal to $b_i \in N$ corresponds to the toric divisor $D_i = \Phi_\omega^{-1}(F_i) \subset X$.

4.2. **Potential function of a Lagrangian torus fibre.** Cho-Oh [9] calculated potentials of Lagrangian torus fibres for Fano toric manifolds and matched them up with mirror Landau-Ginzburg potentials of Givental and Hori-Vafa. Fukaya-Oh-Ohta-Ono [14] studied potentials for general symplectic toric manifolds. Chan [5], Chan-Lau [6] and Chan-Lau-Leung-Tseng [7, 8] have studied the potential functions for semi-positive toric manifolds by establishing an equality between open and closed Gromov-Witten invariants.

Let X be a toric manifold in the previous section. Every free $\mathbb{T}_{\mathbb{R}}$-orbit in X is a fibre of the moment map $\Phi_\omega \colon X \to P(\omega)$ of an interior point in $P(\omega)$, and vice versa. We call it a *Lagrangian torus fibre* of X. For a Lagrangian torus fibre L, we have a homotopy exact sequence:

$$0 \longrightarrow \pi_2(X) \longrightarrow \pi_2(X, L) \xrightarrow{\partial} \pi_1(L) \longrightarrow 0. \tag{30}$$

Let $\beta_i \in \pi_2(X, L)$ denote the class represented by the holomorphic disc $u_i \colon \mathbb{D} \to X$:

$$u_i(z) = [c_1, \dots, c_{i-1}, c_i z, c_{i+1}, \dots, c_m], \quad |z| \leq 1 \tag{31}$$

where $[c_1, \dots, c_m] \in X$ is a point on the Lagrangian L (thus $c_i \neq 0$ for all i). The class β_i intersects with toric divisors as

$$\beta_i \cdot D_j = \delta_{ij}.$$

The relative homotopy group $\pi_2(X, L)$ is an abelian group freely generated by the classes $\beta_1, \dots, \beta_m$ and the toric divisors $D_1, \dots, D_m$ define a dual basis of $H^2(X, L)$. Under the identification:

$$\pi_2(X) \cong H_2(X; \mathbb{Z}) \cong \mathbb{L}, \quad \pi_2(X, L) \cong H_2(X, L; \mathbb{Z}) \cong \mathbb{Z}^m, \quad \pi_1(L) \cong N$$

the exact sequence (30) above can be identified with the fan sequence (26), i.e. $\partial\beta_i = b_i$. The *Maslov index*

$$\mu \colon \pi_2(X, L) \longrightarrow \mathbb{Z}$$

is given by the intersection with $2(D_1 + \cdots + D_m) \in H^2(X, L)$ [9, Theorem 5.1].

We consider the potential function (Definition 2.1) of a Lagrangian torus fibre $L \subset X$. As before, let $\mathcal{M}_1(\beta)$ denote the moduli space of bordered stable maps to (X, L) in the class $\beta \in \pi_2(X, L)$ with one boundary marked point.

Proposition 4.1. *Suppose that $c_1(X)$ is semi-positive. Then $\mathcal{M}_1(\beta)$ is empty for all β with $\mu(\beta) \le 0$. If $\mathcal{M}_1(\beta)$ is non-empty for β with $\mu(\beta) = 2$, then $\beta = \beta_i + d$ for some i and $d \in \mathrm{NE}(X)_{\mathbb{Z}}$ such that $\langle c_1(X), d\rangle = 0$.*

Proof. Let β be a class of a bordered stable map to (X, L). By the classification of holomorphic discs by Cho-Oh [9], we find that β is of the form:

$$\beta = \sum_{i=1}^{m} k_i \beta_i + d \tag{32}$$

for some $k_i \ge 0$ and $d \in \mathrm{NE}(X)_{\mathbb{Z}}$. Here $\sum_{i=1}^m k_i\beta_i$ is the degree of disc components and d is the degree of sphere bubbles. Hence $\mu(\beta) = 2\sum_{i=1}^m k_i + 2\langle c_1(X), d\rangle \ge 0$. We claim that $(k_1, \dots, k_m) = 0$ implies $\mu(\beta) \ge 4$. If $(k_1, \dots, k_m) = 0$, a bordered stable map of class β is the union of a *constant* disc and sphere bubbles. In this case, at least one non-trivial sphere component has to touch L. Let d_1 be the degree of a non-trivial sphere component touching L and let d_2 be the degree of the remaining sphere bubbles. Then $d = d_1 + d_2$ with $d_1, d_2 \in \mathrm{NE}(X)_{\mathbb{Z}}$. Since D_i is disjoint from L, we have $\langle D_i, d_1\rangle \ge 0$. Since $d_1 \ne 0$, we have $\sum_{i=1}^m \langle D_i, d_1\rangle \ge 1$. Also it is impossible that $\sum_{i=1}^m \langle D_i, d_1\rangle = 1$ since d_1 gives the relation $\sum_{i=1}^m \langle D_i, d_1\rangle b_i = 0$ in N. Thus

$$\mu(\beta) = 2\langle c_1(X), d\rangle \ge 2\sum_{i=1}^m \langle D_i, d_1\rangle \ge 4.$$

The claim follows. Consequently, $\mu(\beta) \le 2$ implies $(k_1, \dots, k_m) \ne 0$. The proposition follows easily. □

In particular, the potential function of a Lagrangian torus fibre (Definition 2.1) is well-defined for a semi-positive toric manifold.

Remark 4.2. Fukaya-Oh-Ohta-Ono [14] defined the potential function of a Lagrangian torus fibre even without semi-positivity assumption. They defined virtual cycles and open Gromov-Witten invariants $n_\beta \in \mathbb{Q}$ for all β with $\mu(\beta) = 2$ using $\mathbb{T}_{\mathbb{R}}$-equivariant perturbations, see [14, Lemmata 11.2, 11.5, 11.6, 11.7]. In general, since every effective stable disc class β is of the form (32), the potential W lies in the completed group ring:

$$\mathbb{Q}[\![(\mathbb{Z}_{\ge 0})^m + \mathrm{NE}(X)_{\mathbb{Z}}]\!] \subset \Lambda^{\mathrm{op}}$$

where $\mathrm{NE}(X)_{\mathbb{Z}}$ is regarded as a subset of $\mathbb{Z}^m$ via the second arrow in the fan sequence (26). Notice that $(\mathbb{R}_{\ge 0})^m + \mathrm{NE}(X)$ is a strictly convex cone.

Example 4.3 ([9]). When $\beta = \beta_i$, the moduli space $\mathcal{M}_1(\beta_i)$ consists of holomorphic discs of the form (31) and $\mathrm{ev}\colon \mathcal{M}_1(\beta_i) \cong L$; moreover all such discs are Fredholm regular [9, Theorem 6.1]. Therefore we have $n_{\beta_i} = 1$.

We write

$$z^\beta = z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m} \in \mathbb{Q}[H_2(X, L; \mathbb{Z})]$$

for $\beta = k_1\beta_1 + \cdots + k_m\beta_m$. Also we write

$$q^d = q_1^{\langle p_1, d\rangle} q_2^{\langle p_2, d\rangle} \cdots q_r^{\langle p_r, d\rangle} \in \mathbb{Q}[H_2(X; \mathbb{Z})]$$

for $d \in H_2(X;\mathbb{Z})$, where $p_1, \dots, p_r$ is the nef integral basis of $H^2(X;\mathbb{Z}) \cong \mathbb{L}^\vee$ we chose in §4.1. Note that we have a natural inclusion of the group rings:

$$\mathbb{Q}[H_2(X;\mathbb{Z})] \hookrightarrow \mathbb{Q}[H_2(X,L;\mathbb{Z})].$$

By this we identify q^d with z^d; in co-ordinates:

$$q^d = z^d = z_1^{\langle D_1,d\rangle} z_2^{\langle D_1,d\rangle} \cdots z_m^{\langle D_m,d\rangle} \quad \text{or} \quad q_a = \prod_{i=1}^m z_i^{m_{ai}} \tag{33}$$

where (m_{ai}) is the divisor matrix in (29). Using these notations and Proposition 4.1, we can write the potential function of (X,L) in the following form when $c_1(X)$ is semi-positive.

Definition 4.4. Let X be a semi-positive toric manifold. We present the potential function W of a Lagrangian torus fibre in the form:

$$W = w_1 + \cdots + w_m$$

where $w_i = f_i(q) z_i$ and

$$f_i(q) = \sum_{d \in \mathrm{NE}(X)_{\mathbb{Z}} : \langle c_1(X), d\rangle = 0} n_{\beta_i + d} q^d.$$

We call $f_i(q)$ the *correction term*. This decomposition of W is parallel to Definition 2.2.

Note that we have $f_i(q) = 1 + O(q)$ by Example 4.3. The correction term $f_i(q)$ was denoted by $1 + \delta_i(q)$ in [7]. When X is Fano, all the correction terms are 1 and

$$W = z_1 + \cdots + z_m.$$

This is the result of Cho-Oh [9]. By the *fan polytope*, we mean the convex hull of the ray vectors $b_1, \dots, b_m \in N_{\mathbb{R}}$. In the proof of [7, Corollary 4.12], Chan-Lau-Leung-Tseng showed the following:

Proposition 4.5 (Chan-Lau-Leung-Tseng [7])**.** *Let $f_i(q)$ be the correction terms of the potential of a semi-positive toric manifold X. If the vector b_i is a vertex of the fan polytope of X, then $f_i(q) = 1$.*

4.2.1. *Open-closed moduli space.* We explain that the potential W of a Lagrangian torus fibre can be interpreted as a formal function on the *open-closed moduli space* introduced below.

The *closed moduli space* $\mathfrak{M}_{\mathrm{cl}}$ of X is defined to be:

$$\mathfrak{M}_{\mathrm{cl}} = \{\exp(-\omega + iB) \in \mathbb{L}^\vee \otimes \mathbb{C}^\times \,:\, \omega, B \in \mathbb{L}^\vee \otimes \mathbb{R},\ \omega \in C_X\}.$$

This is also called the *complexified Kähler moduli space*. The nef basis $p_1, \dots, p_r$ of $\mathbb{L}^\vee \cong H^2(X;\mathbb{Z})$ in §4.1 defines $\mathbb{C}^\times$-valued co-ordinates $(q_1, \dots, q_r)$ on $\mathfrak{M}_{\mathrm{cl}} \subset \mathbb{L}^\vee \otimes \mathbb{C}^\times$.

The *open-closed moduli space* $\mathfrak{M}_{\mathrm{opcl}}$ is defined to be the set of triples (q, L, h) such that

- a closed moduli $q = \exp(-\omega + iB) \in \mathfrak{M}_{\mathrm{cl}}$;
- a Lagrangian torus fibre $L = L_\eta = \Phi_\omega^{-1}(\eta)$ at $\eta \in P(\omega)^\circ$;
- a class $h \in H^2(X, L; U(1))$ which maps to $\exp(iB) \in H^2(X; U(1))$.

When the B-field vanishes $B = 0$, the class h defines a $U(1)$-local system on L via the exact seqence:

$$0 \longrightarrow H^1(L;U(1)) \longrightarrow H^2(X,L;U(1)) \longrightarrow H^2(X;U(1)) \longrightarrow 0$$

Let $\eta = (\eta_1, \dots, \eta_m) \in \mathbb{R}^m$ be the co-ordinates of η and write $h = (h_1, \dots, h_m)$ using the identification $H^2(X,L;U(1)) \cong (S^1)^m$; and set

$$z_i := \exp(-\eta_i) h_i. \tag{34}$$

The parameter $z = (z_1, \dots, z_m)$ here determines $\eta_i \in \mathbb{R}$, $h_i \in S^1$ by polar decomposition; then η determines ω by the condition $\eta \in P(\omega)^\circ$ (as $\omega = \kappa(\eta)$) and h determines $\exp(iB)$. Thus z determines a point of $\mathfrak{M}_{\text{opcl}}$. We have:

$$\mathfrak{M}_{\text{opcl}} \cong \left\{ z = (z_1, \dots, z_m) \in (\mathbb{C}^\times)^m \ : \ |z_i| < 1 \text{ for all } i, \ \kappa_{\mathbb{C}^\times}(z) \in \mathfrak{M}_{\text{cl}} \right\}$$

where $\kappa_{\mathbb{C}^\times} \colon (\mathbb{C}^\times)^m \to \mathbb{L}^\vee \otimes \mathbb{C}^\times$ is the third arrow of the divisor sequence (27) tensored with $\mathbb{C}^\times$. A point $z = (z_1, \dots, z_m)$ of the right-hand side parametrizes

- a closed moduli $q = \exp(-\omega + iB) = \kappa_{\mathbb{C}^\times}(z)$;
- a Lagrangian torus fibre $L = L_\eta$ at $\eta = (-\log|z_1|, \dots, -\log|z_m|) \in P(\omega)^\circ$;
- a class $h = (z_1/|z_1|, \dots, z_m/|z_m|) \in H^2(X, L_\eta)$ which is a lift of $\exp(iB)$.

We regard W as a formal function on $\mathfrak{M}_{\text{opcl}}$ via these co-ordinates $(z_1, \dots, z_m)$. The open-closed moduli is fibred over $\mathfrak{M}_{\text{cl}}$:

$$\pi \colon \mathfrak{M}_{\text{opcl}} \to \mathfrak{M}_{\text{cl}}, \quad z \mapsto \kappa_{\mathbb{C}^\times}(z).$$

By pulling-back the co-ordinates $q_1, \dots, q_r$ by π, we obtain the same relation between z_i and q_a as in (33). The fibre $\mathfrak{M}_{\text{opcl},q} = \pi^{-1}(q)$ has the structure of an $(M_{\mathbb{R}}/M) \cong (S^1)^n$-bundle over $P(\omega)^\circ$ via the map:

$$\mathfrak{M}_{\text{opcl},q} \to P(\omega)^\circ, \quad (z_1, \dots, z_m) \mapsto \eta = (-\log|z_1|, \dots, -\log|z_m|).$$

This is a torus fibration dual to the moment map $\Phi_\omega \colon X \to P(\omega)$; we can view it as a mirror of (X, q).

Proposition 4.6. *Via the co-ordinates $(z_1, \dots, z_m)$ on $\mathfrak{M}_{\text{opcl}}$, the potential function of a Lagrangian torus fibre is identified with the following formal sum of functions on $\mathfrak{M}_{\text{opcl}}$:*

$$W(q, L, h) = \sum_{\beta \in \pi_2(X,L) \colon \mu(\beta)=2} n_\beta h(\beta) e^{-\int_\beta \omega}$$

where $q = \exp(-\omega + iB)$.

Proof. For $\beta = \beta_i$, we have (see [9, Theorem 8.1])

$$h(\beta_i) = \frac{z_i}{|z_i|}, \quad \int_{\beta_i} \omega = \eta_i = -\log|z_i|$$

and thus $h(\beta_i) e^{-\int_{\beta_i} \omega} = z_i$ (cf. (34)). Therefore $h(\beta) e^{-\int_\beta \omega} = z^\beta$ for every β. $\square$

Remark 4.7. When $B = 0$, the term $h(\beta)$ is the holonomy along the loop $\partial\beta \in \pi_1(L)$ of the $U(1)$-local system associated to h. This matches with the usual interpretation. In general, this term cannot be interpreted just as holonomy.

Fukaya-Oh-Ohta-Ono [13, Theorem 2.32] showed that the Jacobi algebra of the potential function restricted to the fibre $\mathfrak{M}_{\text{opcl},q} = \pi^{-1}(q)$ is isomorphic to the quantum cohomology ring of (X, q) in a certain q-adic sense.

4.3. **Seidel elements for toric varieties and Givental's mirror transformation.** We review our previous computation [19] relating Seidel elements for toric varieties to Givental's mirror transformation [17]. Let X be a toric manifold from §4.1 with $c_1(X)$ semi-positive.

4.3.1. *Seidel elements associated to the* $\mathbb{C}^\times$*-actions fixing toric divisors.* For each toric divisor D_j of X, we can associate a $\mathbb{C}^\times$-action ρ_j on X rotating around D_j. It is given by:

$$\rho_j(\lambda)\colon [x_1, \dots, x_m] \longmapsto [x_1, \dots, \lambda^{-1} x_j, \dots, x_m], \qquad \lambda \in \mathbb{C}^\times.$$

The toric divisor $D_j = \{x_j = 0\}$ is the maximal fixed component of this action. Let E_j denote the associated bundle of this $\mathbb{C}^\times$-action and let S_j denote the corresponding Seidel element. We also write $S_j = q_0 \widetilde{S}_j$ with $\widetilde{S}_j \in QH^*(X)$ following Definition 2.7. Using the Seidel representation (see Remark 2.9), McDuff-Tolman [24] showed the following multiplicative relations in $QH(X)[q^{-d} : d \in \mathrm{NE}(X)_\mathbb{Z}]$:

$$\prod_{j=1}^{m} \widetilde{S}_j^{\langle D_j, d\rangle} = q^d \qquad \text{for } d \in H_2(X;\mathbb{Z}). \tag{35}$$

4.3.2. *Givental's mirror theorem.* Givental [17] introduced the two cohomology-valued functions

$$I(y,z) = e^{\sum_{i=1}^r p_i \log y_i/z} \sum_{d \in \mathrm{NE}(X)_\mathbb{Z}} \prod_{i=1}^{m} \left(\frac{\prod_{k=-\infty}^{0} (D_i + kz)}{\prod_{k=-\infty}^{\langle D_i, d\rangle} (D_i + kz)} \right) y^d$$

$$J(q,z) = e^{\sum_{i=1}^r p_i \log q_i/z} \left(1 + \sum_j \sum_{d \in \mathrm{NE}(X)_\mathbb{Z} \setminus \{0\}} \left\langle \frac{\phi_j}{z(z-\psi)} \right\rangle_{0,1,d}^{X} \phi^j q^d \right)$$

called the I*-function* and the J*-function* respectively. Here we used a nef basis $\{p_1, \dots, p_r\} \subset H^2(X)$ in §4.1 and write

$$q^d = q_1^{\langle p_1, d\rangle} \cdots q_r^{\langle p_r, d\rangle}, \qquad y^d = y_1^{\langle p_1, d\rangle} \cdots y_r^{\langle p_r, d\rangle},$$

and $\{\phi_j\}$ and $\{\phi^j\}$ are mutually dual basis of $H^*(X)$. The variables $y = (y_1, \dots, y_r)$ are called *mirror co-ordinates*, i.e. co-ordinates of the complex moduli of the mirror Landau-Ginzburg model. Givental [17] showed the following *mirror theorem*:

Theorem 4.8 ([17]). *We have* $I(y,z) = J(q,z)$ *under a change of coordinates of the form* $\log q_i = \log y_i + g_i(y)$, $i = 1, \dots, r$, $g_i(y) \in \mathbb{Q}[\![y_1, \dots, y_r]\!]$ *with* $g_i(0) = 0$. *The functions* $g_i(y)$ *here are uniquely determined by the asymptotics:*

$$I(y,z) = e^{\sum_{i=1}^r p_i \log y_i/z} \left(1 + \sum_{i=1}^{r} g_i(y) \frac{p_i}{z} + o(z^{-1}) \right).$$

The change of co-ordinates is called *mirror transformation* (or *mirror map*).

4.3.3. *Batyrev elements and Seidel elements.* In [19], we introduced *Batyrev elements* $\widetilde{D}_j$, $j = 1, \dots, m$. They are defined by

$$\widetilde{D}_j := \sum_{a=1}^{r} m_{aj} \tilde{p}_a, \quad \tilde{p}_a := \sum_{b=1}^{r} \frac{\partial \log q_b}{\partial \log y_a} p_b.$$

Note that $\widetilde{D}_j$ is an element corresponding to the vector field $\sum_{a=1}^{r} m_{aj} y_a \partial/\partial y_a$ whereas the genuine divisor class D_j corresponds to the vector field $\sum_{a=1}^{r} m_{aj} q_a \partial/\partial q_a$ (see (29)). Batyrev elements are determined by, and determine the Jacobi matrix $(\partial \log q_b / \partial \log y_a)$ of the mirror transformation. Using Givental's mirror theorem, we find that the Batyrev elements satisfy the multiplicative relations (see [19, Proposition 3.8])

$$\prod_{j=1}^{m} \widetilde{D}_j^{\langle D_j, d\rangle} = y^d \qquad d \in H_2(X; \mathbb{Z})$$

in the quantum cohomology ring. These are very similar to the multiplicative relations (35) of Seidel elements, but note that co-ordinates q are replaced with mirror co-ordinates y. Moreover, the Batyrev elements satisfy the following *linear relations*:

$$\sum_{i=1}^{m} c_j \widetilde{D}_j = 0 \quad \text{whenever} \quad \sum_{i=1}^{m} c_j D_j = 0. \tag{36}$$

The linear relations are obvious from the definition. These multiplicative and linear relations show that $\widetilde{D}_j$ satisfy the relations of Batyrev's quantum ring [4]. It turns out that the Seidel elements are multiples of the Batyrev elements.

Theorem 4.9 ([19, Theorem 1.1])**.** *Let $g_0^{(j)}(y)$ be the following hypergeometric series in mirror co-ordinates:*

$$g_0^{(j)}(y_1, \dots, y_r) = \sum_{\substack{\langle c_1(X), d\rangle = 0, \langle D_j, d\rangle < 0 \\ \langle D_i, d\rangle \ge 0 \text{ for all } i \ne j}} \frac{(-1)^{\langle D_j, d\rangle} (-\langle D_j, d\rangle - 1)!}{\prod_{i \ne j} \langle D_i, d\rangle !} y^d. \tag{37}$$

Then under the mirror transformation we have

$$\widetilde{S}_j = \exp\left(-g_0^{(j)}(y)\right) \widetilde{D}_j.$$

Conversely, one can recover the Batyrev elements from the Seidel elements in the following way.

Theorem 4.10 ([19, Theorem 1.2])**.** *Given the Seidel elements $\widetilde{S}_1, \dots, \widetilde{S}_m$, the Batyrev elements $\widetilde{D}_j \in H^*(X) \otimes \mathbb{Q}[\![q_1, \dots, q_r]\!]$, $j = 1, \dots, m$ are uniquely characterized by the following conditions:*

- *(a) $\widetilde{D}_j = H_j \widetilde{S}_j$ for some $H_j \in \mathbb{Q}[\![q_1, \dots, q_r]\!]$;*
- *(b) $\widetilde{D}_j = \widetilde{S}_j$ if b_j is a vertex of the fan polytope;*
- *(c) $\widetilde{D}_j$ satisfy the linear relations* (36).

In particular, the Seidel elements determine the mirror transformation $q \mapsto y$ and the functions $g_0^{(j)}(y)$.

4.4. **Correction terms of potential functions and Seidel elements.** Chan-Lau-Leung-Tseng [7] gave a conjecture relating the correction terms of the potential function and the Seidel elements for a semi-positive toric manifold.

Conjecture 4.11 ([7, Conjecture 5.2])**.** *For a semi-positive toric manifold, the correction term $f_j(q)$ of the potential function (Definition 4.4) coincides with $\exp(g_0^{(j)}(y))$ in Theorem 4.9 under mirror transformation.*

Originally Chan-Lau-Leung-Tseng [7] proved this conjecture under the convergence assumption for W using an isomorphism [13] of Jacobi ring and quantum cohomology. Recently they gave an alternative proof [8] which does not require the convergence assumption. They identified open Gromov-Witten invariants with certain closed Gromov-Witten invariants of the associated bundle E_i' given by the inverse $\mathbb{C}^\times$-action ρ_i^{-1}. They used the fact that a bordered stable map to (M, L) with boundary class $b_i \in N \cong H_1(L)$ can be completed to a holomorphic sphere in the associated bundle E_i'. This is closely related to the fact that the central fibre $\overline{\mathcal{E}}_0$ of the closing in §3.1 is the union of the two associated bundles E and E' which correspond to mutually inverse $\mathbb{C}^\times$-actions.

4.5. **Degeneration formula for toric manifolds.**

Proposition 4.12. *Assumption 3.18 holds for a pair (X, L) equipped with the $\mathbb{C}^\times$-action ρ_j around the prime toric divisor D_j we considered in §4.3.*

Proof. The statement (i) is shown in Proposition 4.1 and (ii), (iii) are obvious. To verify the statement (iv), it is enough to show that every stable map $u\colon C \to E_j$ representing a class $\sigma \in H_2^{\rm sec}(E_j)$ with $\langle c_1^{\rm vert}(E_j), \sigma\rangle = -1$ is contained in $\bigcup_{i=1}^m \widehat{D}_i$, where

$$\widehat{D}_i = D_i \times (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times$$

is a toric divisor of E_j. Let $C = \bigcup C_\alpha$ be an irreducible decomposition of C. If $u_*[C_\alpha]$ is a section class, we have $\langle c_1^{\rm vert}(E_j), u_*[C_\alpha]\rangle \geq -1$ by (3) and the semi-positivity of $c_1(X)$. If $u_*[C_\alpha]$ is not a section class, $u(C_\alpha)$ is contained in a fibre X and we have $\langle c_1^{\rm vert}(E_j), u_*[C_\alpha]\rangle = \langle c_1(X), u_*[C_\alpha]\rangle \geq 0$ again by the semi-positivity. Since $\langle c_1^{\rm vert}(E_j), \sigma\rangle = -1$, we have

$$\left\langle c_1^{\rm vert}(E_j), u_*[C_\alpha]\right\rangle = \begin{cases} -1 & \text{if } u(C_\alpha) \text{ is a section;} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $u(C) \not\subset \bigcup_{i=1}^m \widehat{D}_i$. Then we can find a component C_α such that $u(C_\alpha)$ is not a point and $u(C_\alpha) \not\subset \bigcup_{i=1}^m \widehat{D}_i$. Then $\langle \widehat{D}_i, u_*[C_\alpha]\rangle \geq 0$ for all i. Note that $\sum_{i=1}^m \widehat{D}_i$ is the Poincaré dual of $c_1^{\rm vert}(E_j)$. By the above calculation we see that $\langle \widehat{D}_i, u_*[C_\alpha]\rangle = 0$ for all i and $u(C_\alpha)$ is contained in a certain fibre X. Then $\langle D_i, u_*[C_\alpha]\rangle = 0$ for all i. A homology class $d \in H_2(X)$ satisfying $\langle D_i, d\rangle = 0$ for all i is zero. This is a contradiction. □

Recall from Remark 4.2 that the potential function $W = W(z_1, \dots, z_m)$ of a toric manifold X is an element of

$$R := \mathbb{Q}[\![\mathrm{NE}(X)_\mathbb{Z} + (\mathbb{Z}_{\geq 0})^m]\!] \subset \Lambda^{\rm op}.$$

We also set

$$K := \mathbb{Q}[\![\mathrm{NE}(X)_\mathbb{Z}]\!] \subset \Lambda.$$

Then R is a K-algebra (cf. (33)). For $f \in R$, we write (following notation in Theorem 3.20):

$$df = \left(z_1 \frac{\partial f}{\partial z_1}, \dots, z_m \frac{\partial f}{\partial z_m} \right) \in \mathbb{Z}^m \otimes R \cong H_2(X, L) \otimes R.$$

In other words,

$$dz^\beta = \beta \otimes z^\beta$$

for $\beta \in H_2(X, L)$.

We apply Theorem 3.20 to the $\mathbb{C}^\times$-action ρ_j rotating around D_j. Note that the k-th term w_k of the potential W in Definition 4.4 corresponds to the boundary class $b_k \in N \cong H_1(L)$ and $w_k = W_{b_k}$ in the notation of Definition 2.2. Since the Seidel element $\widetilde{S}_j$ in §4.3 belongs to $H^2(X) \otimes K$, we have $\widetilde{S}_j^{(0)} = 0$ and $\widetilde{S}_j = \widetilde{S}_j^{(2)}$. By Proposition 4.12, we can define the lift

$$\widehat{S}_j \in H^2(X, L) \otimes K$$

of $\widetilde{S}_j = \widetilde{S}_j^{(2)}$ as in Definition 3.19. The class λ of an S^1-orbit on L is $-b_j \in H_1(L)$ and the maximal disc class α_0 is β_j. Hence we obtain:

Theorem 4.13. *Assume that the degeneration formula (Conjecture 3.17) holds for (X, L) equipped with the $\mathbb{C}^\times$-action ρ_j around the toric divisor D_j (see §4.3). Then we have*

$$\langle \widehat{S}_j, dw_k \rangle = \delta_{jk} z_j. \tag{38}$$

In particular, we have $\langle \widehat{S}_j, dW \rangle = z_j$.

4.5.1. *Example.* Consider the second Hirzebruch surface $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1})$, a compactification of $\mathcal{O}_{\mathbb{P}^1}(-2)$. The divisor matrix (29) is:

$$(m_{ai}) = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

The column vectors give toric divisors classes D_1, D_2, D_3, D_4. Here D_1 is the ∞-section, D_2 is the zero-section (-2 curve) and D_3, D_4 are fibres. The potential function has been calculated by Auroux [3], Fukaya-Oh-Ohta-Ono [16] and Chan-Lau [6]:

$$W = z_1 + (1 + q_1) z_2 + z_3 + z_4.$$

Therefore we have

$$\begin{bmatrix} dw_1 \\ dw_2 \\ dw_3 \\ dw_4 \end{bmatrix} = \begin{bmatrix} z_1 & 0 & 0 & 0 \\ 0 & (1 - q_1) z_2 & q_1 z_2 & q_1 z_2 \\ 0 & 0 & z_3 & 0 \\ 0 & 0 & 0 & z_4 \end{bmatrix}$$

where we used $q_1 = z_2^{-2} z_3 z_4$ (see (33)) and $d(q_1 z_2) = [0, -q_1 z_2, q_1 z_2, q_1 z_2]$. Assuming the degeneration formula (38), we obtain

$$\left[\widehat{S}_1, \widehat{S}_2, \widehat{S}_3, \widehat{S}_4 \right] = [D_1, D_2, D_3, D_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-q_1} & -\frac{q_1}{1-q_1} & -\frac{q_1}{1-q_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is compatible with the calculations of $\widetilde{S}_j$ by McDuff-Tolman [24] and González-Iritani [19].

4.6. **Kodaira-Spencer map.** Recall from Definition 4.4 that $w_i = f_i(q) z_i$ for some $f_i(q) \in K$. We have (using (33))

$$z_i \frac{\partial w_j}{\partial z_i} = \left(\delta_{ij} + z_i \frac{\partial f_j(q)}{\partial z_i} \right) z_j = \left(\delta_{ij} + \sum_{a=1}^{r} m_{ai} q_a \frac{\partial f_j}{\partial q_a}(q) \right) z_j \in K z_j.$$

Therefore we have an isomorphism of K-modules:

$$\mathfrak{ks} \colon H^2(X,L) \otimes K \xrightarrow{\cong} \bigoplus_{j=1}^{m} K z_j, \quad D_i \longmapsto \left(z_i \frac{\partial w_1}{\partial z_i}, \dots, z_i \frac{\partial w_m}{\partial z_i} \right).$$

The degeneration formula (38) says that $\mathfrak{ks}(\widehat{S}_i) = z_i$. For $\varphi \in H^1(L) = M$, we have

$$\begin{aligned} \mathfrak{ks}(\delta\varphi) &= \bigoplus_{j=1}^{m} \sum_{i=1}^{m} \langle \varphi, b_i \rangle z_i \frac{\partial w_j}{\partial z_i} = \bigoplus_{j=1}^{m} \sum_{i=1}^{m} \langle \varphi, b_i \rangle \left(z_i \delta_{ij} f_j(q) + z_i z_j \frac{\partial f_j(q)}{\partial z_i} \right) \\ &= \bigoplus_{j=1}^{m} \langle \varphi, b_j \rangle \, w_j \in \bigoplus_{j=1}^{m} K z_j. \end{aligned}$$

where $\delta \colon H^1(L) \cong M \to H^2(X,L) \cong \mathbb{Z}^m$ is a coboundary map. Hence $\mathfrak{ks}$ induces an isomorphism

$$\mathrm{ks} \colon H^2(X) \otimes K \xrightarrow{\cong} \bigoplus_{j=1}^{m} K z_j \Big/ \left\langle \textstyle\bigoplus_{j=1}^{m} \langle \varphi, b_j \rangle \, w_j \ : \ \varphi \in M \right\rangle_K .$$

This satisfies $\mathrm{ks}(\widetilde{S}_i) = [z_i]$. Set $B_j := f_j(q) \widetilde{S}_j$, $j = 1, \dots, m$. Then $\mathrm{ks}(B_j) = f_j(q)[z_j] = [w_j]$, $j = 1, \dots, m$ satisfy the linear relations

$$\sum_{j=1}^{m} \langle \varphi, b_j \rangle \, [w_j] = 0$$

for all $\varphi \in M$. Consequently,

- $B_j = f_j(q) \widetilde{S}_j$;
- $f_j(q) = 1$ if b_j is a vertex of the fan polytope (Proposition 4.5);
- B_j, $j = 1, \dots, m$ satisfy the linear relations (by the injectivity of ks).

By the characterization of the Batyrev elements (see Theorem 4.10), we know that $B_j = \widetilde{D}_j$, i.e. $f_j(q) = \exp(g_0^{(j)}(y))$. This shows the conjecture of Chan-Lau-Leung-Tseng:

Theorem 4.14. *Assume that the degeneration formula (Conjecture 3.17) holds for* (X, L) *equipped with the* $\mathbb{C}^\times$*-actions* ρ_j, $j = 1, \dots, m$ *in* §*4.3. Then Conjecture 4.11 holds.*

Remark 4.15. Via the natural map $\bigoplus_{j=1}^{m} K z_j \to R$, the map ks induces the so-called Kodaira-Spencer map (cf. the discussion at the end of §3.3.3):

$$\mathrm{KS} \colon H^2(X) \otimes K \to \mathrm{Jac}(W)$$

where the Jacobi algebra $\mathrm{Jac}(W)$ is defined to be

$$\mathrm{Jac}(W) := R / R \langle H^1(L), dW \rangle.$$

Then we have $\mathrm{KS}(\widetilde{S}_i) = [z_i]$ and $\mathrm{KS}(\widetilde{D}_i) = [w_i]$. In other words, the Seidel elements are the inverses of $[z_i]$ and the Batyrev elements are the inverses of $[w_i]$.

4.7. **Consistency check: computing equivariant Seidel elements.** Here we give a consistency check concerning Chan-Lau-Leung-Tseng Conjecture 4.11 and our degeneration formula (38). We calculate the lifts $\widehat{S}_j$ of Seidel elements assuming Conjecture 4.11 and (38) and see that the result is compatible with our previous calculation [19]. The lifts $\widehat{S}_j$ here should be viewed as the $\mathbb{T}$-equivariant Seidel elements since $H^2_{\mathbb{T}}(X) \cong H^2(X, L)$.

Lemma 4.16. *Suppose that Conjecture 4.11 holds. Then* $w_i = f_i(q) z_i$, $i = 1, \dots, m$ *satisfy the multiplicative relation*

$$\prod_{j=1}^m w_j^{\langle D_j, d\rangle} = y^d \qquad \text{for all } d \in H_2(X;\mathbb{Z}).$$

In other words, $y_a = \prod_{j=1}^m w_j^{m_{aj}}$, $a = 1, \dots, r$.

Proof. Recall that the Seidel and the Batyrev elements satisfy the multiplicative relations with respect to the quantum product (§4.3):

$$\prod_{j=1}^m \widetilde{D}_j^{\langle D_j, d\rangle} = y^d, \quad \prod_{j=1}^m \widetilde{S}_j^{\langle D_j, d\rangle} = q^d.$$

Hence we have

$$\prod_{j=1}^m f_j(q)^{\langle D_j, d\rangle} = y^d / q^d.$$

Therefore

$$\prod_{j=1}^m w_j^{\langle D_j, d\rangle} = \prod_{j=1}^m \left(f_j(q)^{\langle D_j, d\rangle} z_j^{\langle D_j, d\rangle} \right) = (y^d/q^d) \cdot q^d = y^d.$$

□

Theorem 4.17. *Assume Conjecture 4.11 and the degeneration formula* (38). *The lifts* $\widehat{S}_j$ *of the Seidel elements are given by*

$$\widehat{S}_j = e^{-g_0^{(j)}(y)} \left(D_j - \sum_{i=1}^m D_i \sum_{\substack{c_1(X)\cdot d = 0, D_i \cdot d < 0, \\ D_k \cdot d \geq 0 \text{ for all } k \neq i.}} (-1)^{\langle D_j, d\rangle} \langle D_j, d\rangle \frac{(-\langle D_i, d\rangle - 1)!}{\prod_{k\neq i} \langle D_k, d\rangle !} y^d \right)$$

under the mirror transformation.

Proof. Note that $(dw_1, \dots, dw_m)^{\mathrm{T}}$ can be viewed as the Jacobi matrix between the two coordinate systems $(w_1, \dots, w_m)$ and $(\log z_1, \dots, \log z_m)$ on the open-closed moduli space. The degeneration formula (38) implies that $(z_1^{-1}\widehat{S}_1, \dots, z_m^{-1}\widehat{S}_m)$ is the inverse Jacobi matrix, i.e.

$$z_j^{-1}\widehat{S}_j = \sum_{i=1}^m \frac{\partial \log z_i}{\partial w_j} D_i = w_j^{-1} \sum_{i=1}^m \frac{\partial \log z_i}{\partial \log w_j} D_i$$

in $H^2(X, L)$. Assuming Conjecture 4.11, we have $\log z_i = \log w_i - g_0^{(i)}(y)$. Hence

$$\begin{aligned}\widehat{S}_j &= \frac{z_j}{w_j}\sum_{i=1}^m \left(\delta_{ij} - w_j \frac{\partial g_0^{(i)}}{\partial w_j}\right) D_i \\ &= \exp\left(-g_0^{(j)}(y)\right)\left(D_j - \sum_{i=1}^m \sum_{a=1}^r m_{aj} y_a \frac{\partial g_0^{(i)}}{\partial y_a} D_i\right)\end{aligned}$$

In the second line, we used Lemma 4.16. The conclusion follows. □

Note that we did not use the lifts $\widehat{S}_j$ of the Seidel elements (but used only the original Seidel elements $\widetilde{S}_j$) in the proof of Theorem 4.14.

Remark 4.18. This result is compatible with the calculation of $\widetilde{S}_j$ in our previous paper [19]. Note however that the formula in [19, Lemma 3.17] contains a mistake. It occurred from an erroneous cancellation between the factors $\langle D_j, d\rangle$ in the numerator and $\langle D_j, d\rangle!$ in the denominator.

Remark 4.19. It is not difficult to generalize the computation in [19] to the $\mathbb{T}$-equivariant setting and to check the above computation of $\widehat{S}_j$ without using Conjecture 4.11 and the degeneration formula (38). Since Chan-Lau-Leung-Tseng's conjecture 4.11 was proved by themselves [8], it follows that the degeneration formula (38) holds true in toric case.

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